Semester Report WS05/06 of Sarah Kappes

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Field of Research:	Discrete Mathematics
Topic:	Orthogonal Surfaces
PhD Student	at the program since October 1st, 2003

Field of Research and Results

Let $V \subset \mathbb{N}^d \subset \mathbb{R}^d$ be an antichain with respect to the dominance order.The filter generated by V is $\langle V \rangle = \{\alpha \in \mathbb{R}^d : \alpha \geq v \text{ for some } v \in V\}$. The boundary of $\langle V \rangle$ is an orthogonal surface S_V.

The corners of the surface are called *characteristic points*. The *cp-order* of S_V is the set of characteristic points equipped with the dominance order. This poset captures the most important part of the combinatorial structure of the surface.

We call S_V generic if no two points of V share any coordinates. In this case, the cp-order can be extended to the face-lattice of a simplicial polytope minus one facet, this is a restatement of the Theorem of Scarf, [3]. On the other hand, not all simplicial polytopes have a corresponding cp-order.

This leads to the *Realization question:* Given a (simplicial) d-polytope P, is there a corresponding cp-order of an orthogonal surface in d dimensions?

The other crucial question is which non-generic orthogonal surfaces give rise to polytopal cp-orders. In the general, the cp-order is not graded, does not have the diamond-property and thus does not even satisfy the most basic properties needed to define a face-complex. In the past I tried to find restrictions that are weaker than genericity but still enable us to guarantee some nice properties for the cp-order. This led to the definition of *non-degenerate* and *rigid* surfaces.

For non-degenerate surfaces, I was able to prove the existence of *orthogonal matchings* on the set of characteristic points. The ith orthogonal matching M_i is defined by $(c, d) \in M_i \iff c_j = d_j$ for all $j \neq i$ and $c_i < d_i$. This matching is almost perfect, only one minimum remains unmatched.

For rigid surfaces, the orthogonal matchings satisfy an additional acyclicity-property, this leads to the following proposition:

Proposition 1. *An orthogonal matching on the cp-order of a non-degenerate, rigid orthogonal surface is a Morse-matching.*

In this direction, there are several open questions I would like to solve:

- If the orthogonal surface is non-degenerate and rigid and if the cp-order satisfies the diamond-property, is it a CW-poset?
- When does the cp-order admit recursive coatom ordering? In the generic case, the lexicographic order of the maxima is such an ordering. In the non-generic case, there are examples where the lexicographic order does not work.

Another useful approach with regard to the realizability problem is to find classes of realizable polytopes and identify polytope constructions that maintain realizability. The following results are of that type:

Proposition 2. *If a polytope P is realizable, then the pyramid $pyr(P)$ is also realizable.*

Proposition 3. *All d-polytopes with $d + 2$ vertices are realizable.*

Proposition 4. *The d-dimensional crosspolytope is realizable.*

Except for these and some other special classes, the gap between cp-orders and face-lattices is still wide, and there are many interesting open questions, for example:

- The removal of a facet corresponds to the removal of a vertex in the dual polytope. Therefore, it is not obvious whether the duals of realizable polytopes are always realizable. It would be interesting to find an example where this is not the case - or a construction showing that it is always possible to realize the dual polytope.

Activities

- CGC-Workshop in Hiddensee (September 26th - 28th 2005)
- Attended the Noon-Seminar of the workgroup “Diskrete Mathematik” at TU Berlin
  - Talk “A combinatorial definition for characteristic points”
  - Talk “Realizers for d-polytopes with $d + 2$ vertices”

- Attended the seminar “Partielle Ordnungen” of the workgroup “Diskrete Mathematik” at TU Berlin
- Attended the ”Monday lectures and Colloquia” of the CGC

  Talk “Non-generic orthogonal surfaces” (January 30th 2006)
- Attended the “EuroComb ’05” at TU Berlin
- Longstay at TU Eindhoven in the workgroup “Algorithms”, January 16th-February 28th 2006

Preview

- Blockcourse “Embeddings of Planar Graphs” at TU Berlin
- Workshop on “Geometric and Topological Combinatorics” & Advanced Course “Combinatorial and Computational Geometry: Trends and topics for the future” in Madrid (August 31-September 5, 2006)

References

[1] S. Felsner, S. Kappes: Orthogonal Surfaces, in preparation

[2] M. Joswig, M. Pfetsch: Computing Optimal Morse Matchings, ZIB Report 04-37, arXiv math.CO/0408331

[3] H. E. Scarf: The Computation of Economic Equilibria, vo. 24 of Cowles Foundation Monographs, Yale University Press, 1973.