Semester Report WS 04/05 of Arnold Waßmer

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Field of Research and Results

In we last semester I mainly wrote up my thesis. It will contain three parts:

- A Dual Independence Complex of Paths, Trees and Cycles,
- A Discrete Version of the Borsuk-Ulam Theorem,
- The Symmetries of Cyclic Polytopes.

In this semester a note about box complexes appeared [3]. It was written with the (former) CGC students Péter Csorba, Carsten Lange, and Ingo Schurr.

In the following I will present some details about the dual independence complexes. The independence complex $\mathtt{IC}(G)$ of a graph G is the simplicial complex that is given by the independent sets of G. Thus, the vertex set of $\mathtt{IC}(G)$ is the node set $V(G)$. A set of vertices forms a simplex in $\mathtt{IC}(G)$ if the corresponding node set in G does not contain two neighboring nodes. In general this complex $\mathtt{IC}(G)$ is not pure, this means its inclusion maximal simplices vary in dimension.

For forests T the complex $\mathtt{IC}(T)$ either is contractible or is homotopy equivalent to a sphere, see [4] and [5]. As a consequence, also the links of simplices $\sigma \in \mathtt{IC}(T)$ either are contractible or are homotopy equivalent to a sphere. In the latter case we say that σ is a *spherical independent set* of T. Let $\mathtt{SPH}(T)$ be the poset of all spherical independent sets of a forest T. According to [2] there is a homotopy equivalence between the complex $\mathtt{IC}(T)$ and the order complex of $\mathtt{SPH}(T) \setminus \emptyset$. Let $\mathtt{DIP}(T) := \mathtt{SPH}(T)^{\Delta} \cup \{\hat{0}\}$ be the dual poset of $\mathtt{SPH}(T)$ with an additional minimal element.

I will prove in my thesis that for every forest T this dual independence poset $\mathtt{DIP}(T)$ is a CW-poset, see [1]. This means, that there is a regular cell complex $\mathtt{DIC}(T)$ such that its face poset is $\mathtt{DIP}(T)$. This complex is the *dual independence complex* of T. This name is inspired by polytope theory since dual polytopes also have dual face lattices (see [6]).

For example, there is a tree T such that its independence complex $\texttt{IC}(T)$ is the boundary of the octahedron with one full tetrahedron attached to one of the triangles. Dualizing yields a cell complex $\texttt{DIC}(T)$ which is the boundary complex of a cube. In this sense dualization of independence complexes yields smaller posets and nicer complexes.

To prove that the poset $\texttt{DIP}(T)$ is a CW-poset, one mainly has to show that the ideal below each element is homeomorphic to a sphere. This guarantees that the boundaries of the cells in $\texttt{DIC}(T)$ are homeomorphic to spheres. The boundary of each cell in $\texttt{DIC}(T)$ is isomorphic to the boundary of the complex $\texttt{DIC}(T')$ for some subforest $T' \subset T$. Its easy to see that its independence complex $\texttt{IC}(T')$ is homotopy equivalent to a sphere. Such forests are called *spherical* forests. To understand their combinatorics I studied the facets of their dual independence complexes. These facets correspond to spherical independent sets that are minimal under inclusion.

To comprehend these sets of nodes I classified the nodes of a forest into six different types. I worked out which pairs of types may be neighbors, which may not and which types always have neighbors of a certain types. In spherical forests only four of these six types actually do occur. The set of facets of a spherical forest can be described in the language of these types. Facets are independent sets that either consist of one node of the first or of the second type, or they are certain sets of at least two nodes of the third type. Moreover the dimension of the complex $\texttt{DIC}(T)$ can be obtained by counting how many nodes of which type there are. Finally this classification of the nodes of a forest is a great help for proving that the poset $\texttt{DIP}(T)$ has a rank function and has the diamond property.

By deleting a certain node, one can decompose a big forest T into a subforest T' such that there is a homotopy equivalence $\texttt{IC}(T) \simeq \texttt{IC}(T')$, see [5]. If T is a spherical forest, one can prove that this induces for the dual posets a homeomorphism $\texttt{DIP}(T) \cong \texttt{DIP}(T')$. Iterated application of this decomposition shows that for spherical forests T the dual independence complex $\texttt{DIC}(T)$ is homeomorphic to a cube. Thus its boundary is homeomorphic to a sphere. This is the main argument in the proof that $\texttt{DIP}(T)$ is a CW-poset.

Since this shall be my last semester report I want to thank all members of the graduate program who made this work possible: the coordinators Bettina Felsner and Andrea Hoffkamp, and the speaker of the program Helmut Alt. Especially I want to thank my supervisor Günter M. Ziegler for his initial inspiration for this project and his enduring support. Thank you very much!

References

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