Semester Report WS04/05 of Mathias Schacht

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Field of Research:	Discrete Mathematics
Topic:	Extremal problems for graphs and hypergraphs
Postdoc	at the program since August 2004

Field of Research and Results

Let $\mathcal{F}_0$ be a fixed k-uniform hypergraph. An (integer) $\mathcal{F}_0$*-packing* in a given hypergraph $\mathcal{H}$ is defined to be a set of copies of $\mathcal{F}_0$ in $\mathcal{H}$ which are pairwise edge-disjoint. We denote the set of all copies of $\mathcal{F}_0$ in $\mathcal{H}$ by $\binom{\mathcal{H}}{\mathcal{F}_0}$, and we write $\nu_{\mathcal{F}_0}(\mathcal{H})$ for the maximum size of an $\mathcal{F}_0$-packing in $\mathcal{H}$. The problem of calculating $\nu_{\mathcal{F}_0}(\mathcal{H})$ is NP-complete. In fact this is true even for graphs F_0 that contain a connected component with at least three edges (see [1]). However, the fractional version of that problem can be solved in polynomial time by linear programming. A *fractional* $\mathcal{F}_0$*-packing* in $\mathcal{H}$ is a function

$$\psi\colon \binom{\mathcal{H}}{\mathcal{F}_0} \to [0,1]$$

which satisfies for every $e \in E(\mathcal{H})$

$$\sum \left\{ \psi(\mathcal{F})\colon\ \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0} \text{ and } e \in \mathcal{F} \right\} \leq 1\,. \tag{1}$$

The weight $|\psi|$ of a fractional $\mathcal{F}_0$-packing is $\sum \left\{\psi(\mathcal{F})\colon\ \mathcal{F} \in \binom{\mathcal{H}}{\mathcal{F}_0}\right\}$ and the fractional $\mathcal{F}_0$-packing number $\nu^*_{\mathcal{F}_0}(\mathcal{H})$ for a given $\mathcal{H}$ is $\max_\psi |\psi|$, where the maximum is taken over all fractional $\mathcal{F}_0$-packings in $\mathcal{H}$. Clearly, $\nu^*_{\mathcal{F}_0}(\mathcal{H}) \geq \nu_{\mathcal{F}_0}(\mathcal{H})$. On the other hand, it is desirable to have a lower bound for $\nu_{\mathcal{F}_0}(\mathcal{H})$ in terms of $\nu^*_{\mathcal{F}_0}(\mathcal{H})$.

In [5] this problem problem was addressed for graphs. There the authors showed for every fixed graph F_0 and real $\xi > 0$ that

$$\nu_{F_0}(H) \geq \nu^*_{F_0}(H) - \xi|V(H)|^2 \quad \text{for } |V(H)| \text{ sufficiently large}\,. \tag{2}$$

Owing to (2) the NP-complete problem of determining $\nu_{F_0}(H)$ can be approximated in polynomial time for those graphs for which $\nu_{F_0}(H) = \Omega(|V(H)|^2)$.

Therefore this problem (in the graph case) is another example of an NP-complete problem which has a polynomial time approximation algorithm for an appropriately defined "dense case" . Moreover, the proof of (2) in [5] is constructive and gives a polynomial time algorithm for finding an (integer) F_0-packing in H of size $\nu^*_{F_0}(H) - \xi|V(H)|^2$. The proof was based on the algorithmic version of Szemerédi's Regularity Lemma and the algorithmic version (due to Grable [3]) of the packing theorem of Frankl and Rödl [2].

Subsequently in [4] the analogous inequality of (2) for 3-uniform hypergraphs was proved. The proof in [4] is based on the Regularity Method for 3-uniform hypergraphs and somewhat technical and quite long.

In collaboration with Rödl, Siggers, and Tokushige [7] we found a simpler proof based on an improved version of the Regularity Lemma for hypergraphs [6]. That proof extended to k-uniform hypergraphs.

Theorem 1. *For every integer $k \geq 2$, for every fixed k-uniform hypergraph $\mathcal{F}_0$ and every real $\xi > 0$ there exist n_0 such that for every k-uniform hypergraph $\mathcal{H}$ on $n \geq n_0$ vertices*

$$\nu_{\mathcal{F}_0}(\mathcal{H}) \geq \nu^*_{\mathcal{F}_0}(\mathcal{H}) - \xi n^k .$$

Recently Yuster [8] proved a sufficient condition under which a hypergraph $\mathcal{H}$ admits fractional $\mathcal{F}_0$-decomposition, i.e., a fractional $\mathcal{F}_0$-packing ψ^* which satisfies (1) with equality for every $e \in E(\mathcal{H})$. For a real $0 \leq \gamma \leq 1$ we say a k-uniform hypergraph $\mathcal{H}$ on n vertices is γ-dense if for every $i = 1, \ldots, k-1$

$$\min_{I \in \binom{[n]}{i}} \left|\left\{e \in E(\mathcal{H}) \colon\ e \supset I\right\}\right| \geq \gamma \binom{n-i}{k-i} .$$

Theorem 2 (Yuster [8]). *For every k-uniform hypergraph $\mathcal{F}_0$ there exists an $\alpha > 0$ and some n_0, such that for all $n > n_0$ every k-uniform, $(1-\alpha)$-dense hypergraph $\mathcal{H}$ on n vertices admits a fractional $\mathcal{F}_0$-decomposition.*

The corollary below follows from a combined application of Theorem 1 and Theorem 2.

Corollary 3. *For every k-uniform hypergraph $\mathcal{F}_0$, and for all $\eta > 0$, there exists an $\alpha > 0$ and some n_0, such that for all $n > n_0$ every k-uniform, $(1-\alpha)$-dense hypergraph $\mathcal{H}$ on n vertices admits an (integer) $\mathcal{F}_0$-packing that covers $(1-\eta)|E(\mathcal{H})|$ of the edges.*

Activities

- Jointly with Ehud Friedgut (Jerusalem) and Vojtěch Rödl we taught the **DIMATIA DOCCOURSE 2005** for doctoral students on “Modern Methods in Ramsey Theory” at the Charles University in Prague (01-02/2005).
- Talks “On the Regularity Method for hypergraphs” given at:
  - 4th Annual Workshop on Combinatorics, Geometry, and Computation, Stels (10/2004)
  - DIMACS/DIMATIA/Rényi Working Group on Extremal Combinatorics II, DIMACS Center, Rutgers University, Piscataway (10/2004)
  - Kolloquium über Kombinatorik, Magdeburg
  - Discrete Mathematics Seminar, A. Mickiewicz University, Poznań (11/2004)
  - Graduiertenkolleg Angewandte Algorithmische Mathematik, Technische Universität München (12/2004)
- Talks on “Discrepancy and Eigenvalues of Cayley graphs” given at:
  - SIAM Seminar, Emory University, Atlanta (10/2004)
  - Theoretical Computer Science Seminar, ETH Zürich (01/2005)
- Seminars and Learn- & Workshop of the research group “Algorithms, Structure, Randomness”

Preview

I was granted funding for “One’s own position” by the Deutsche Forschungsgemeinschaft and I will leave the graduate program in the middle of Februar 2005.

References

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[3] D. A. Grable, *Nearly-perfect hypergraph packing is in NC*, Inform. Process. Lett. **60** (1996), no. 6, 295–299.

[4] P. E. Haxell, B. Nagle, and V. Rödl, *Integer and fractional packings in dense 3-uniform hypergraphs*, Random Structures Algorithms **22** (2003), no. 3, 248–310.

[5] P. E. Haxell and V. Rödl, *Integer and fractional packings in dense graphs*, Combinatorica **21** (2001), no. 1, 13–38.

[6] V. Rödl and M. Schacht, *Regular partitions of hypergraphs*, manuscript (45 pages).

[7] V. Rödl, M. Schacht, M. Siggers, and N. Tokushige, *Integer and fractional packings of hypergraphs*, submitted.

[8] R. Yuster, *Fractional decompositions of dense hypergraphs*, submitted.