Semester Report WS04/05 of Sarah Renkl

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Field of Research:	Discrete Mathematics
Topic:	Orthogonal Surfaces
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Research and Results

Orthogonal surfaces

Let $V \subset \mathbb{N}^d \subset \mathbb{R}^d$ be an antichain with respect to the dominance order. Assume that V is in general position, i.e. no two points share a coordinate. The *filter* generated by V is $\langle V \rangle = \{\alpha \in \mathbb{R}^d \mid \alpha \geq v \text{ for some } v \in V\}$. The boundary of $\langle V \rangle$ is an *orthogonal surface* S_V in dimension k.

The *orthogonal triangulation* T_V is a face complex generated by V in the following way: $F \subset V$ is a face of T_V if and only if there is no $v \in V \setminus F$ such that $v \leq \bigvee(F)$, where $\bigvee(F)$ is the component-wise maximum over all $f \in F$.

Faces with two elements are called *edges*, faces with d elements are called *facets*.

As shown in [3], an orthogonal triangulation is the face complex of a simplicial polytope minus one facet. On the other hand, not every simplicial polytope minus any facet can be realized as an orthogonal triangulation.

I have shown the following property of orthogonal triangulations:

Result (Edge-Facet Criterion). *If $\{v, w\} \subset V$ is an edge and $v \vee w = (v_1, \ldots, v_k, w_{k+1}, \ldots, w_d)$, then there are at least $k * (d - k)$ facets containing the edge $\{v, w\}$.*

For $d = 4$, this has some interesting implications. There are two possible types of edges, namely (v_1, v_2, v_3, w_4) and (v_1, v_2, w_3, w_4). For the first type, there are at least three facets containing the edge, for the second, there are at least four.

Since the number of edges of the first type is fixed at $4n - 10$, any additional edge has to be of the second type. A simple counting argument can prove the non-realizability: If a polytope has less than $|E| - (4n - 10)$ edges that are contained in at least four facets each, it is not realizable. For many examples, this criterion is sufficient to prove the non-realizability.

Future work

- There are non-realizable 4-polytopes where every edge is contained in at least four facets, so the edge-facet criterion cannot be applied. I would like to find some other criteria that prove the non-realizability of these examples.
- Interestingly, every *neighborly* 4-polytope minus one facet and up to 9 vertices is realizable if and only if the edge-facet criterion holds. This has been verified by computing the edge-facet incidence numbers for all these examples, which were provided by Frank Lutz.

  I would like to find out if this equivalence is also true for $n \geq 10$.
- Currently, I work on orthogonal surfaces in the more general setting where the antichain V is not in general position. This implies that the orthogonal complex contains non-simplex faces. Many questions remain open here. It is not even certain that all orthogonal complexes are shellable. As a further step, the realization issue can be extended to non-simplicial polytopes.

Pseudo-convex decompositions

Let S be a set of n points in general position in the plane. A *pseudo-convex decomposition of* S is a set of internally disjoint convex polygons and pseudotriangles such that all vertices are in S and the union is the convex hull of S.

For a given point set, we are interested in the minimal size of a decomposition, the *pseudo-convex decomposition number* $\psi_d(S)$. The pseudo-convex decomposition number for all sets of fixed size n is $\psi_d(n) = \max_S \psi_d(S)$

This number mostly depends on the number and size of empty convex subsets of S. With the help of the order type data base ([1]), we obtained the following results ([2]):

Result. *Every set of eight points contains an empty convex pentagon or two disjoint convex quadrilaterals.*

This implies that $\psi_d(8) = 4$

Result. *Every set of eleven points contains an empty convex hexagon or an empty convex pentagon an a disjoint empty convex quadrilateral.*

This implies that $\psi_d(11) = 6$
A Divide-and-Conquer argument yields the following general upper bound:

$$\psi_d(n) = \frac{\psi_d(k) + 1}{k - 1}$$

The best current upper bound is achieved with $k = 11$, namely

$$\psi_d(n) \leq \frac{\psi_d(11) + 1}{11 - 1} n = \frac{7}{10} n$$

We presume that this bound is not tight. A lower bound to the pseudo-convex decomposition number is $\psi_d(n) \geq \frac{2}{3}n$. It is an open question whether this is also the correct asymptotic upper bound.

Activities

- CGC-Workshop in Stels, October 4th-7th, 2004

  Talk: "Orthogonal Surfaces - A realization criterion"

- Symposium Diskrete Mathematik at the ETH Zurich, October 7th-8th, 2004

- Longstay at TU Eindhoven, September 6th - November 30th, 2004

- Attended the lecture "I/O-efficient algorithms" at TU Eindhoven

- Attended the Noon Seminar of the Algorithms-Group at TU Eindhoven

  Talk "Orthogonal Surfaces"

- Attended the Monday lectures and colloquia of the CGC

  Colloquium Talk "Convex-Pseudo Decompositions" December 6th, 2004

- Visit TU Eindhoven December 15th - 18th, 2004

Preview

- European Workshop in Computational Geometry at the TU Eindhoven, March 9th-11th, 2005

  Talk: "On Pseudo-Convex Decompositions, Partitions, and Coverings"

References

[1] Aichholzer, Aurenhammer, Krasser: "Enumerating order types for small point sets with applications", Proc. 17th Annu. ACM Symposium Comput. Geom. (2001), 11-18.

[2] Aichholzer, Huemer, Renkl, Speckmann, Toth: "On Pseudo-Convex Decompositions, Partitions, and Coverings", extended abstract to appear at EWCG 2005

[3] Bayer, Peeva, Sturmfels: "Monomial Resolutions", Math. Res. Lett. 5 (1998), no. 1-2, 31-46.