Semester Report WS 04/05 of Stephan Hell

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Topic:	Topological Methods in Combinatorics and Geometry
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Current work and Perspectives

This semester I mainly worked in three fields. In the following three paragraphs I give more details on my work. The order of the paragraphs reflects the chronical order of my work on them. The first two go back to previous semesters so less background is given there. During my "long stay" in Prague, I have started studying intersection patterns of finite families under the supervision of Jiří Matoušek. The KAM institute in Prague is a very welcoming and lively place: Nice working atmosphere and a big number of active people – guests and locals – headed by Jaroslav Nešetřil and Jiří Matoušek. I was very glad being there.

Topological Tverberg. First I worked on the open case of the topological Tverberg theorem and on the related Sierkma's conjecture following different approaches, see my previous semester reports for background and results. Recall that all proofs of topological Tverberg use the following scheme: The non–existence of a Tverberg partition implies the existence of some G–equivariant map $f : X \to Y$, where G is a finite group acting on some tailor–made topological spaces X and Y. Finally, results from algebraic topology imply the non–existence of this map, e. g. Borsuk–Ulam is used for $G = \mathbb{Z}_2$. In the prime power case deeper results on characteristic classes are needed, see [3]. I went through the proof in [3] to see where prime powers are needed. This is related to facts on irreducible complex representations of $(\mathbb{Z}_p)^k$. The existence of the map f in the general case is well–known, but that does not immediately disprove topological Tverberg. Constructing such a map first, and then perturbing it a little bit might lead to a counter–example. Calculating the top obstruction cocycle during the construction would also give new insight.

Moreover, I have refined my test programs that define drawings of complete graphs randomly in a first step, and then count the number of Tverberg partitions by brute force. Up to now, I have checked geodesic drawings on the sphere, rectilinear drawings, and drawings with piecewise linear edges consisting of three pieces. All computer results confirm Sierkma's conjecture. A solution of topological Tverberg via the above approach would lead to non–trivial lower bounds in the general case. Following Torsten Schöneborn's results, I looked at the following questions on drawings of complete graphs: Does moving one edge and fixing all others maintain the property of having a winding partition? Do new partitions come up when old ones vanish? The answer to the second question is no, as observed by my advisor. A positive answer to the first would prove the smallest unproven case of topological Tverberg.

Cubical 4–polytopes. Last semester, I studied f–vectors of cubical 4–polytopes with a *small* number of vertices starting with the classification of Blind & Blind. The question whether there is a cubical 4–polytope on 34 vertices remained open. $f_0 = 34$ is the last candidate for the gap as well–known constructions lead to cubical 4–polytopes for any even $f_0 \geq 36$, see my semester report SS 04. Together with Axel Werner from our group at TU Berlin a computer project was started for enumerating cubical 4–polytopes. My input to this project is mainly theoretical giving conditions for shelling sequences leading to cubical 4–polytopes inspired by Blind & Blind's work. The setup of this project is more general involving shelling sequences for other families of 4–polytopes. Until present, other families are studied by Axel Werner. When I will be back at TU Berlin we will continue this classification based on the coming results from the computer project.

Intersection patterns. Starting point of my project with Jiří Matoušek was the paper [2] of Matoušek et al. on (p, q)–theorems for finite families of sets arising in geometry. The theory involved is in some parts quite technical, for a basic introduction see [5]. Historically, the paper [4] of Gil Kalai can be seen as a starting point for the results of [2]. This paper contains as a key result the *Upper Bound Theorem for d–collapsible complexes (UBT).* The UBT implies a Fractional Helly Theorem (FHT) for finite families of convex sets in $\mathbb{R}^d$ via their nerve complexes. FHT leads then to a (p, q)–theorem for finite families of convex sets. In [2], this proof scheme is generalized to an abstract machinery for showing a (p, q)–theorem given a FHT for a collection of finite families of sets. Moreover, observing that d–Leray complexes are d–collapsible the authors of [2] generalize the above FHT to finite families of sets having a d–Leray nerve complex. Using the above machinery they also generalize the above (p, q)–theorem to finite families $\mathcal{F}$ of sets in $\mathbb{R}^d$ such that intersections $\cap\mathcal{H}$ are either empty or contractible for all $\mathcal{H} \subset \mathcal{F}$. The aim of my project is to see whether a FHT holds for more general families having a topologically complexer intersection pattern. The simplest way to increase this topological complexity is to increasing the number of components. In this case, I have rediscovered a FHT for families of sets being the union of a bounded number of convex sets with slightly better constants than in [1]. This can be obtained by generalizing the proof of a weaker version of the FHT for finite families of convex sets, see [5]. The method proving the above UBT is known as Algebraic Shifting, it uses mainly linear algebra. For completeness, Jiří Matoušek proved a FHT for finite families of sets with bounded VC–dimension, e. g. sets defined by a finite number of polynomial inequalities of bounded degree in $\mathbb{R}^d$. Gil Kalai has asked the question: Is there a homological analog of VC–dimension?

Activities

- Annual workshop of the CGC, October 4–7, Stels, Switzerland
- Attended DMV Symposium *Diskrete Mathematik*, October 7–8, Zürich, Switzerland
- "Long stay" for 5 months at KAM institute, Prague, visiting Jiří Matoušek, September 25, 2004 – February 25, 2005
- Talk at *Noon seminar* of KAM institute, October 25, Prague
- Talk *On the number of Tverberg partitions in the prime power case* at Workshop organized by Rade Živaljević , October 28–31, Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade
- Attended lecture *Combinatorial and Additive Number Theory* of Oriol Serra and Jaroslav Nešetřil, KAM, Prague
- Attended lectures of *Modern Methods in Ramsey Theory*, January 24 – February 24, DOCCOURSE PRAGUE 2005

References

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[2] N. Alon, G. Kalai, J. Matoušek, and R. Meshulam, *Transversal numbers for hypergraphs arising in geometry*, Adv. in Appl. Math. **29** (2003), pp. 79–101.

[3] M. de Longueville, *Notes on the topological Tverberg theorem*, Discr. Math. **247** (2002), pp. 271–297.

[4] G. Kalai, *Intersection patterns of convex sets*, Isr. J. Math. **48** (1984), pp. 161–174.

[5] J. Matoušek, *Lectures on Discrete Geometry*, Graduate Texts in Mathematics 212, Springer, 2002.