Semester Report WS02/03 of Arnold Waßmer

Name:	Arnold Waßmer
Supervisor:	Prof. Dr. Günter M. Ziegler
Field of Research:	Discrete and Combinatorial Geometry
Topic:	Topology in DCG
PhD Student	at the program since June 1, 2001

Field of Research

The starting point of topological combinatorics was László Lovász' proof of Kneser's conjecture in 1978. He translated the problem of finding lower bounds for graph coloring into the topological terms of determining the connectivity of a space. Using the Borsuk-Ulam theorem he settled an open problem about the Kneser graphs (see [4]).

The topological space he definded is the neighborhood complex. Many variations of this complex were invented in the sequel, for example the so called *Lovász complex* and the *box complex* with their variants (see [5]). The youngest child in this family is the *homomorphism complex*, which also was defined by László Lovász. This complex $\mathtt{Hom}(G, H)$ is a regular cell complex given for any two graphs G and H. Its vertices are the graph homomorphisms from G to H. The cells of of $\mathtt{Hom}(G, H)$ correspond to sets of homomorphisms that are similar in a certain sense (see [1]). This complex is a generalization of the box complex since $\mathtt{Hom}(K_2, H)$ is homotopy equivalent to $\mathtt{Box}(H)$.

The connectivity of the homcomplex $\mathrm{Hom}(G, H)$ for a family of graphs G has consequences for the chromatic number of the second graph H: Eric Babson and Dmitry Kozlov [1] proved the following conjecture by Lovász: If for any r the hom complex $\mathtt{Hom}(C_{2r+1}, H)$ is k-connected then $\chi(H) \geq 4 + k$.

In the proof of this theorem some special homomorphism complexes are examined: the complexes $\mathtt{Hom}(G, K_n)$. The links of their cells turn out to be determined by the independence complex $\mathtt{Ind}(G)$. Thus I studied the independence complex for a better understanding of the homomorphism complex.

The vertices of the independence complex $\mathtt{Ind}(G)$ are the nodes of G; the independent sets of G form the simplices of $\mathtt{Ind}(G)$. If the graph is a tree T then $\mathtt{Ind}(T)$ is either contractible or is homotopy equivalent to a sphere (see [3]). In this context one may ask for a simple criterion to determine if $\mathtt{Ind}(T)$ is contractible (see [2]).

Results

In joint work with Günter M. Ziegler I could give a criterion to determine if $\texttt{Ind}(T)$ is contractible. For example if T contains two leaves of distance three then T is contractible.

If the independence complex $\texttt{Ind}(T)$ is not contractible then it is homotopy equivalent to a sphere, as already mentioned. Except in a few special cases $\texttt{Ind}(G)$ is a non pure simplicial complex, in particular then it is not homeomorphic to a sphere.

For paths and cycles G we could dualize these complexes in a sense analogous to polytope duality. The vertices of the dual independence complex $\texttt{DInd}(G)$ are the inclusion maximal simplices of $\texttt{Ind}(G)$. The cells of $\texttt{DInd}(G)$ correspond to those simplices of $\texttt{Ind}(G)$ with a non contractible link. The dual independence complex $\texttt{DInd}(G)$ has a smaller face poset than the primal complex $\texttt{Ind}(G)$. It turns out that they are homotopy equivalent; $\texttt{DInd}(G) \simeq \texttt{Ind}(G)$. Moreover for paths P the dual complexes $\texttt{DInd}(P)$ are *homeomorphic* to balls or to spheres. In other words dualization made the complexes not only smaller but nicer.

Preview

The next step is to generalize this concept to trees. The main goal is to dualize the homomorphism complex $\texttt{Hom}(G, K_n)$ for trees and cycles G.

Activities

- Lecture *A quantified version of the Borsuk-Ulam Theorem* at *Kolloquium über Kombinatorik,* Magdeburg, November 14 – 15 , 2003
- Lecture *The homomorphism complex* at *Workshop on Computational Geometry*, Neustrelitz, October 1, 2003
- Lecture *On the independence complexes of paths, trees, and cycles* in the CGC colloquium, at Humbold Universität, Berlin January 12, 2004

References

[1] Eric Babson and Dmitry N. Kozlov, *Topological obstructions to graph colorings.* arxiv:math.CO/0305300, 2003.

[2] Richard Ehrenborg and Gábor Hetyei, *The topology of the independence complex.* preprint, 2003.

[3] Dmitry N. Kozlov, *Complexes of directed trees*, J. Comb. Theory Ser. A, **88** (1999), pp. 112–122.

[4] László Lovász, *Kneser's conjecture, chromatic number, and homotopy*, Journal of Combinatorial Theory, Series A, **25** (1978), pp. 319–324.

[5] Jiří Matoušek and Günter M. Ziegler, *Topological lower bounds for the chromatic number: A hierarchy.* preprint, to appear in Jahresbericht der DMV, 2004.