Semester Report WS03/04 of Ares Ribó Mor

Supervisor:	Prof. Dr. Günter Rote
Field of Research:	Geometry and Combinatorics
Topic:	Self-Touching Configurations
	Map Foldability and Rigidity Theory
	Spanning trees of Planar Graphs
PhD Student	at the program since February 1, 2002

Field of Research and Results

This semester I visited three months the research group of Prof. Dr. Jiří Matoušek at Charles University in Prague, as a part of my long term exchange.

The topics that I studied with more effort during the last months are:

Self-Touching Configurations

Last semester we generalized the *Maxwell-Cremona theorem* to self-touching configurations. We gave an explicit correspondence between the set of stresses of a planar self-touching configuration and the set of three-dimensional polyhedral terrains with *jump discontinuities* (vertical facets) that project onto it. We showed that, while the lifting is unique up to the addition of an affine function, the projection is not unique since the kernel of the lifting is non trivial: a set of non-zero forces lifts to the flat polyhedra.

We have used this tool to attack again the question of if every planar self-touching configuration can be perturbed into a simple configuration. If this is true, it would also mean that self-touching chains and cycles are always infinitesimally flexible (this is also one of our questions).

Studying the properties of the three-dimensional polyhedral terrain given by this correspondence, we have shown that a lifting of a self-touching configuration with no stressed bars can not have a minimum point in the interior. We have also shown that the minimum of such a configuration can not be found between self-touching parallel bars. If we have non stressed bars, the only possibility then is that the outer face has the minimum height zero and the polyhedral terrain is convex with positive heights. We still do not know if this is possible or not. If the answer is no, it means that the only possible lifting is the flat one, and this would prove that the only self-touching stress in equilibrium is the zero stress, hence a perturbation would always exist.

Number of Spanning Trees of a Planar Graph

I started to work on this topic in Prague, in contact with my advisor Prof. Rote by e-mail. We want to bound the number T of spanning trees of a planar graph with n vertices.

For lower bounds, we have introduced a new method based on transfer matrices for enumerating T for recursively constructible families of graphs. We have implemented the method with Maple for several families of graphs, for example the triangulated lattice. From the larger eigenvalue of the generated transfer matrices, we obtain the asymptotic value of T for this families. The obtained values coincide with the analytic results found by Shrock and Wu in [4]. The interesting thing is that they consider lattices with periodic boundary conditions, hence non planar. We think we can prove that the eigenvalues of the transfer matrices with and without periodic boundary conditions are the same. This would prove that the lower bound 5.029^n is also valid for planar graphs.

For upper bounds, we can show that $T \leq 5.333^n$. This is easy considering the upper bound for k-regular graphs given by McKay, Chung, and Yau, with $k = 3$ (the dual of our planar triangulated graph is a 3-regular graph with $2n - 4$ vertices). Prof. Rote has upper bounds for graphs without triangles ($T \leq 4^n$) and graphs without triangles and quadrilaterals (($T \leq 2.924^n$). This bounds are not tight and we think they can be improved.

The motivation is in connection with trying to embed 3-polytopes on small integer grids using the Maxwell-Cremona Theorem.

Another related open question is to study how an equilibrium stress at the interior vertices of a configuration without triangles and quadrilaterals can be extended to an equilibrium everywhere.

Polygons with the (p, q)-Property

Prof. Matoušek suggested me the following problem in the context of art galleries: A polygon $X \in \mathbb{R}^2$ with holes has the (p, q)-property if for any p points in X, at least q of them can be seen from a single point of X. It is known that a polygon with the $(3, 3)$-property is star-shaped (due to Krasnoselski), that $(6, 5)$-property implies that at most 4 guards can see all of the polygon X, and that in general, if a polygon has the (p, q)-property with $p > q \geq 2\lceil \frac{p}{3} \rceil + 1$, then a small number of points suffice to guard all the polygon X. It is also known that polygons with the $(2q - 1, q)$-property,

$q \geq 3$, do not ensure any finite number of guards. In this sense, for example, the $(5,3)$-property is bad: the polygon can need a large number of points to be guarded. The question is: " which p, q ensure a small number of guards?". For the case $(2,2)$, I found and example of a polygon that needs arbitrarily many guards. We studied the case $(4,3)$, which is a particular case of $(2,2)$, but at the moment we are still not able to say if $(4,3)$ implies a small number of guards or not. Also other cases like $(n,4)$, for $n \geq 5$, are open.

Folding Grids of Paper

We want to determine the complexity of deciding whether a $m \times n$ grid of paper with a mountain-valley assignment (i.e. with given folding direction of each crease) can be folded or not. Even the $2 \times n$ case is difficult. Studying local conditions for flat foldability, last year I wrote an algorithm, based on a known algorithm for folding strips of stamps, to fold a given $2 \times n$ grid. During my visit in Prague I attended the *Workshop on Graph Homomorphisms*. I exposed the folding problem there. Discussing with other students we realized that the number of possible foldings of a given $2 \times n$ grid with mountain-valley assignment can be exponential. Thus it makes no sense to look at all possible foldings given by the algorithm; I must find a *canonical way of folding*, such that if a given pattern can be folded, then it can be folded canonically. In Prague I have been working on this canonical folding. Now I have some ideas, and although I thought for a while that I already had a characterization of flat foldable $2 \times n$ grids, the complexity of the problem is still open.

Activities

- Attended the *Monday Lectures and Colloquia* of the Graduate Program. Presentation of the talk *The Maximum Number of Spanning Trees of a Planar Graph*, February 2th, 2004.

- Attended the *Mittagsseminar Theoretische Informatik* at Freie Universität Berlin. Presentation of the talk *The Maximum Number of Spanning Trees of a Planar Graph*, January 29th, 2004.

- Stay of three months at the Department of Applied Mathematics of the Charles University Prague, at the research group of Prof. Jiří Matoušek and Prof. Jan Kratochvil, from September 7th to November 31th, 2003.

- Attended to the weekly Noon Seminar of the Department of Applied Mathematics of the Charles University Prague. Presentation of the talk *A Generalisation of the Maxwell-Cremona Theorem for Self-Touching Configurations*, October 23th, 2003.

- *EUROCOMB 2003*, Prague, September 8–12, 2003.

- *Workshop on Graph Homomorphisms and related topics HOMONOLO 2003*, Nova Louka (Czech Republic), September 15–19, 2003 . Presentation of the talk *Folding a* $2 \times n$ *grid of paper.*

- *Workshop on Combinatorics, Geometry and Computation*, CGC–Program, Neustrelitz, September 28–October 1, 2003. Presentation of the talk *A Generalisation of the Maxwell-Cremona Theorem for Self-Touching Configurations.*

- *Fall School on Computational Geometry*, Neustrelitz, October 2–4, 2003.

- Subreferee for *SODA 2004*.

References

[1] R. Connelly, E. Demaine, G. Rote, *Infinitesimally locked self-touching linkages with applications to locked trees*,"Physical Knots: Knotting, Linking, and Folding Geometric Objects in $\mathbb{R}^3$". Contemporary Mathematics 304, American Mathematical Society 2002, 287–311.

[2] J. Matousek. *Lectures on discrete geometry*, Vol. 212 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2002.

[3] J. Richter-Gebert, *Realization spaces of polytopes*, chapter 13. Lecture Notes in Mathematics **1643**, Springer–Verlag Berlin Heidelberg 1996.

[4] R. Shrock, F. Y. Wu, *Spanning trees on graphs and lattices in d dimensions*, J.Phys. A33, 2000, 3881–3902.