Semester Report WS03/04 of Sarah Renkl

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Field of Research:	Discrete Mathematics
Topic:	Orthogonal Surfaces and Schnyder-Colorings
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Field of Research

W. Schnyder introduced a labelling of the angles in a planar triangulated graphs to show that planar graphs have a dimension ≤ 3, [3].

Due to its nice combinatorial properties, this labelling can be used to obtain to a convex embedding of a planar triangulated graph with n vertices in a grid of small size, [4]. It furthermore induces an edge-labelling in three colors, partitioning the inner edges into three spanning trees. Both notions, the labelling of the angles and of the edges, can be generalized to all 3-connected planar graphs, [1].

A different point of view on the structure of these labelled graphs is offered by the *orthogonal surfaces* in dimension 3. Let $V \subset N^3 \subset R^3$ be an antichain in the dominance order. The *filter* generated by V is the set

$$\langle V \rangle = \{\alpha \in \mathbb{R}^3 : \alpha \geq v \text{ for some } v \in V\}$$

The boundary of this set is called an *orthogonal surface*. There is a bijection between certain orthogonal surfaces (*rigid*, with three special outer vertices) and Schnyder-colored 3-connected planar maps, [2].

While the definition of the Schnyder-colorings can only be applied to 3-connected planar graphs, the definition of an antichain, a filter and an orthogonal surface can of course be generalized to arbitrary dimensions.

The main field of my research so far is the task of characterizing the objects (for example the graphs) that arise from orthogonal surfaces of dimension 4, and to check whether there is an analogue to the Schnyder-labelling in higher dimensions.

Since the graphs that relate to 3-dimensional orthogonal surfaces are exactly the graphs of 3-dimensional polytopes (by the Theorem of Steinitz, the 3-connected and planar graphs), one might ask whether the higher dimensional structures are also related to polytopes.

All graphs that can be embedded in a 4-dimensional orthogonal surface have order dimension at most 4. Therefore, certainly not all graphs of 4-polytopes have such an embedding, since the complete graphs K_n have order dimension > 4 for all $n > 12$ and can occur as subgraphs of the edge graphs of 4-polytopes.

I examined how many 1-, 2- and 3-dimensional faces can occur in the orthogonal surface of an n-element antichain.

If the elements of the antichain are in general position, two elements cannot lie in the same orthogonal plane, that means that they differ in each coordinate. That implies that all faces are simplices. In this case, the minimal number of faces is reached if and only if the graph is *stacked*, having $4n - 6$ edges, and then there is a corresponding 4-polytope, which is a stacked polytope. There are other embeddable triangulations with up to $4n + \frac{n^2}{4} + O(n)$ edges.

For the labellings, that means that a coloring of the edges with 4 colors does generally not result in a decomposition into 4 spanning trees. However, a four-color-labelling of the angles with nice properties can be defined.

To visualize some aspects of the structures decribed above, I wrote a program that represents a 4-dimensional orthogonal surface as a sequence of 3-dimensional orthogonal surfaces.

Many questions remain open so far. In the future, I will hopefully resolve some of the following:

- Are there small structures that can not be embedded and thus indicate whether the induced graphs etc. are polytopal? For example, I would like to find a proof that Barnette's topological 3-sphere is not realizable on an orthogonal surface of dimension 4.

- Are there other notions that can be generalized? In three dimensions, the projection of an orthogonal surface onto a plane naturally leads to a tiling of the plane with 4-gons. In four dimensions, there could be a similar tiling of 3-space with combinatorial cubes.

- In 3 dimensions, an the orthogonal surface of an arbitrary antichain might contain degeneracies. To ensure that the surface produces a graph, these cases have to be excluded. In higher dimensions, there might be even more possibilities of degenerate situations. How can these be characterized? Which of them have to be forbidden to maintain a good (graphs etc.) structure?

Activities

- CGC-Workshop in Neustrelitz (September 28th - October 1st)
- CGC Fallschool on Computational Geometry in Neustrelitz (October 2nd - October 4th)
- Attended the lecture "Topology" of Professor G. M. Ziegler
- Attended the "Monday lectures and Colloquia" of the CGC
- Attended the "Kolloquium on Combinatorics" in Magdeburg (November 14th-15th)

Preview

- CGC-Doccourse in Prague, January 2004-March 2004
- COMBSTRU-Workshop in Bordaux (April 1st - 3rd 2004)

References

[1] S. Felsner, Convec Drawings of Planar Graphs and the Order Dimension of 3-Polytopes, Order 18 (2001), pp. 19-37.

[2] S. Felsner, Geodesic Embedding and Planar Graphs, Order 20 (2003), pp. 135-150.

[3] W. Schnyder, Planar graphs and poset dimension, Order 5 (1989), pp. 323-343.

[4] W. Schnyder, Embedding planar graphs on the grid, in Proc. 1st. ACM-SIAM Sympos. Discrete Algorithms, 1990, pp. 138-148.