Semester Report WS 03/04
of Andreas Paffenholz

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Field of Research:	Discrete Geometry
Topic:	Flag Vectors of 4-Polytopes
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Field of Research and Results

I pursued three different lines of research in this semester. Most of the time I worked on some new families of polytopes for which we can find polytopal realizations for their E-construction [Paf04]. These polytopes also have some nice properties with respect to the flag vector classification problem (cf. [Zie02]). From the end of the last semester and until November I completed, together with A. Björner, J. Sjöstrand and G.M. Ziegler, our work on "Bier spheres" [BSPZ03]. In between I spent also some time on the continuation of my work on cubical and cocubical polytopes together with Carsten Lange [LP04]. In the following I will describe these results in more detail.

In [Paf04] – of which I hopefully will have finished a first draft before I leave to my block course in Prague – I will describe a new family of polytopes to which the E-construction, which I have described with G.M. Ziegler in [PZ03], is applicable. I prove that, whenever one has a product of two polytopes such that there is a polytopal realization of the E-construction applied to the two factors, together with some additional conditions, then also the E-construction of the product has a polytopal realization.

The idea to consider these polytopes did not come from the aim to provide new examples for the E-construction, but from an entirely different source. G. Gévay and J. Bokowski described a family of self-dual 3-spheres and asked for for polytopal realizations of these. I proved that all their spheres can be obtained by applying the E-construction to the product of two polygons and showed that these products satisfy the conditions mentioned in the previous paragraph. Thus all the spheres do have a polytopal realization. These polytopes provide nice and natural examples where the geometric and combinatorial symmetry groups differ. Among other things I constructed a very simple 4-parameter group of realizations for the "24-cell" that are pairwise not projectively equivalent.

In view of the classification problem of flag vectors of 4-polytopes, D. Eppstein, G. Kuperberg, and G.M. Ziegler introduced the "fatness" parameter $F := \frac{f_1+f_2-20}{f_0+f_3-10}$ and provided examples of polytopes with fatness around 5.

They also proved that fatness is unbounded for regular CW spheres, while for polytopes it is not known whether fatness is bounded or not. The polytopes that I construct in [Paf04] provide examples of polytopes with fatness arbitrarily close to 6. This is best possible for polytopes obtained from the E-construction applied to simple polytopes. Polytopes of high fatness are interesting as they lie in a rather "unexplored area" in the cone of possible flag vectors (cf. [Zie02]).

My work with A. Börner, J. Sjöstrand and G.M. Ziegler was in a slightly different direction (presented in [BSPZ03]). We consider in this paper a construction that provides a simple way to produce very many new lattices from a given one. This construction is a generalization of a construction of Th. Bier and has great similarity with the E-construction from [PZ03]: Given is a lattice L and a proper ideal I in this lattice. We define a new lattice $\mathrm{Bier}(L, I)$ in the following way.

- The new elements are intervals $[x, y]$ in L such that $x \in I$ and $y \notin I$.
- We define a new rank function on this set by reversed inclusion of intervals in L.

The construction preserves the length and the properties of being graded and Eulerian. As in the above case with the E-construction we can prove that the construction applied to face lattices of spheres gives again face lattices of spheres. Thus with our construction we obtain very many regular CW spheres. In fact we obtain many more than there can be polytopes. Thus most of these spheres cannot have a polytopal realization.

In the special case of a face lattice of a simplex and an ideal therein we prove the following fact.

Theorem. *The Bier sphere* $\mathrm{Bier}(I, L)$ *is shellable for every proper ideal* I *in the boolean lattice* L. ☐

We also provide an explicit shelling order for these spheres.

A third project connected with the construction of polytopes with special properties is a joint work with Carsten Lange on k-cubical and h-cocubical d-polytopes and $(d-1)$-spheres [LP04]. We basically prove the following theorem

Theorem. *There are no k-cubical and h-cocubical $(d-1)$-spheres for $k+h \geq d+1$.* ☐

We provide two independent proofs for this fact: One topological proof that constructs the universal cover of such a sphere, and one metric proof, that constructs a metric of nonpositive curvature on such a sphere. For 4-polytopes we provide another metric proof based on a suitable triangulation of the polytope. With this last approach we hope to get some better insight into the problem whether there exist 2-cubical and 2-cocubical polytopes (in any dimension $d \geq 4$). A negative answer to this questions would produce some interesting restrictions on the flag vector.

This semester will for me be ended with a block course in Prague from Mid-January to the beginning of March. I will attend both courses offered there. The first one is by Peter Cameron on "Permutation groups, structures, and polynomials" and the second is by Micha Sharir on "Arrangements in Computational and Combinatorial Geometry".

Further Directions

There are still several open problems connected with those two constructions that I hope to solve in the next time. For the E-construction this includes:

- I would like to have more and more general families of polytopes for which the E-constructions have polytopal realizations.
- The construction of realizations for products of polytopes at present works satisfyingly only for products of polygons, though I know several examples in higher dimensions.
- The product construction in its present form only produces realizations for the parameter $t = d - 2$. It should be possible to extend the conditions such that they work for the whole parameter range.
- From the method to construct polytopal realizations of the E-construction applied to products I have some new ideas of how to attack the problem of finding polytopal realization for other families of polytopes. These might also lead to examples with a fatness above six.

For the Bier-construction I will try to extend the results in the following direction:

- Shellability doesn't seem to be restricted to Boolean lattices. There should be a much larger class of lattices for which we can prove shellability of its Bier spheres.

The big remaining question in the project on cubical/cocubical polytopes is

- whether there exist 2-cocubical and 2-cubical polytopes.
  A bit more restrictive is
- whether there are k-cubical and h-cocubical d-polytopes for $k + h = d$.

Activities

I attended the following lectures, seminars, workshops, and conferences:

- Attended the class on "Discrete Differential Geometry" of A. Bobenko, TU Berlin, WS 03/04

- Lectures and Colloquia of the CGC
- CGC Annual Workshop, Neustrelitz, September 28 – October 1, 2003
- Fall School on Computational Geometry, October 1 – 4, 2003
- Workshop "Polyhedral Surfaces," July 25 – 29, St Petersburg
- Kolloquium über Kombinatorik, November 14 – 15, 2003, Magdeburg

I gave a talk on the following occasions:

- Talk on "New Constructions for Polytopes", September 29, 2003, Workshop on "Polyhedral Surfaces", St Petersburg.
- Talk on "Bier Spheres", October, 1, 2003, CGC Annual Workshop, Neustrelitz
- Talk on "Polytopes derived from Products," November 2003, Colloquium of the CGC, FU Berlin
- Talk on "Polytopes derived from Products," November 2003, Kolloquium über Kombinatorik, Magdeburg

Forthcoming activities are

- block course "Doc course Berlin-Prague 2004" in Prague from Mid-January to the beginning of March.
- "long-stay" in Zurich starting in April or May

References and Publications

[BSPZ03] Anders Börner, Jonas Sjöstrand, Andreas Paffenholz, and Günter M. Ziegler, *Bier spheres and posets*, submitted, available at `arXiv:math.CO/0311356`, November 2003, 11 pages.

[LP04] Carsten Lange and Andreas Paffenholz, *Cubical and cocubical polytopes*, in preparation, January 2004.

[Paf04] Andreas Paffenholz, *New polytopes derived from products*, in preparation, January 2004.

[PZ03] Andreas Paffenholz and Günter M. Ziegler, *The E_t-Construction for Lattices, Spheres and Polytopes*, submitted, available at `arXiv:math.MG/0304492`, April 2003, 20 pages.

[EKZ03] David Eppstein, Greg Kuperberg, and Günter M. Ziegler, *Fat 4-polytopes and fatter 3-spheres*, Discrete Geometry: in honour of W. Kuperberg's 60th birthday (A. Bezdek, ed.), Pure and Applied Mathematics. A series of Monographs and Textbooks, vol. 253, Marcel Dekker, Inc., 2003, `arXiv:math.CO/0204007`, pp. 239–265.

[Zie02] Günter M. Ziegler, *Face Numbers of 4-Polytopes and 3-Spheres*, Proceedings of the ICM 2002, vol. III, 2002, pp. 625–634.