Semester Report WS03/04 of Daniela Kühn

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Field of Research:	Discrete Mathematics
Topic:	Graph Theory
Postdoc	at the program since 1 October 2003

Fields of current Research & Results

Unbalanced complete bipartite minors

Let $d(s)$ be the smallest number such that every graph of average degree greater than $d(s)$ contains the complete graph K_s as minor. The existence of $d(s)$ was first proved by Mader [8]. Kostochka [4] and Thomason [11] independently showed that the order of magnitude of $d(s)$ is $s\sqrt{\log s}$. Later, Thomason [12] was able to prove that $d(s) = (\alpha + o(1))s\sqrt{\log s}$, where $\alpha = 0.638\dots$ is an explicit constant.

Recently, Myers and Thomason [10] extended the results of [12] from complete minors to H minors for arbitrary dense (and large) graphs H. However, not much is known for sparse graphs H. One partial result in this direction was obtained by Myers [9]: he showed that every graph of average degree at least $t + 1$ contains a $K_{2,t}$ minor. This is best possible as he observed that for all positive ε there are infinitely many graphs of average degree at least $t + 1 - \varepsilon$ which do not contain a $K_{2,t}$ minor. More generally, Myers [9] conjectured that for every fixed s there exists a positive constant C such that for all t every graph of average degree at least Ct contains a $K_{s,t}$ minor. Together with Deryk Osthus, I proved the following strengthened version of this conjecture [6].

Theorem 1 *For every $0 < \varepsilon < 10^{-16}$ there exists a number $t_0 = t_0(\varepsilon)$ such that for all integers $t \geq t_0$ and $s \leq \varepsilon^7 t/\log t$ every graph of average degree at least $(1+\varepsilon)t$ contains $K_s + \overline{K}_t$ as a minor.*

(Here $K_s + \overline{K}_t$ denotes the graph which is obtained from $K_{s,t}$ by adding all edges between the vertices in the vertex class of size s.) Theorem 1 is essentially best possible in two ways. Firstly, the complete graph K_{s+t-1} shows that up to the error term εt the bound on the average degree cannot be reduced. Secondly, Theorem 1 breaks down if we try to set $s \geq 18t/\log t$.

Moreover, if $t/\log t = o(s)$ then even a linear average degree (as in the conjecture of Myers) no longer suffices to force a $K_{s,t}$ minor.

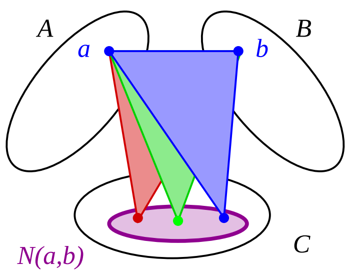
For the case when s is much smaller than logarithmic in t, Kostochka and Prince [5] obtained more precise upper and lower bounds on the average degree required to force $K_s + \overline{K_t}$ as a minor. (They proved these results slightly later but independently of us.)

Matchings in hypergraphs

One of the classical results in Graph Theory is the theorem of Hall from 1935 which provides a necessary and sufficient condition for the existence of a perfect matching in a bipartite graph. For hypergraphs there is no analogue of this result – up to now only partial results are known. For example, Conforti et al. [2] extended Hall's theorem to so-called balanced hypergraphs. A sufficient condition for matchability for a certain class of hypergraphs was also proved by Haxell [3]. Moreover, there are many results about the existence of almost perfect matchings in hypergraphs which are pseudo-random in some sense, see e.g. [1].

A simple corollary of Hall's theorem for graphs states that every bipartite graph with vertex classes A and B of size n whose minimum degree is at least $n/2$ contains a perfect matching. Together with Deryk Osthus, I proved an analogue of this result for hypergraphs [7]. For simplicity, I will only describe the situation for 3-uniform hypergraphs here, but the results carry over to r-uniform hypergraphs.

So let us consider a 3-partite 3-uniform hypergraph H with vertex classes A, B and C where $|A| = |B| = |C| = n$. Let E denote the set of hyperedges of H. Thus the elements of E are triples abc with $a \in A$, $b \in B$ and $c \in C$. One way to define the minimum degree of H is the following. Given vertices $x, y \in A \cup B \cup C$, the *neighbourhood* $N(x,y)$ *of* x *and* y in H is the set of all those vertices z which form a hyperedge together with x, y, i.e. for which $xyz \in E$.



The *minimum degree* $\delta_2(H)$ is then defined to be the minimum $|N(x,y)|$ over all pairs x, y which lie different vertex classes of H.

Theorem 2 *There exists an integer* n_0 *such that every* 3*-uniform* 3*-partite hypergraph* H *whose three vertex classes have size* $n \geq n_0$ *and whose minimum degree* $\delta_2(H)$ *is at least* $n/2 + n^{2/3}$ *has a perfect matching.*

Theorem 2 is best possible up to the error term $n^{2/3}$. Suprisingly, a simple argument already shows that a significantly smaller minimum degree guarantees a matching which covers all but at most 3 vertices of H:

Theorem 3 *There exists an integer* n_0 *such that every* 3*-uniform* 3*-partite hypergraph* H *whose three vertex classes have size* $n \geq n_0$ *and whose minimum degree* $\delta_2(H)$ *is at least* $n/3$ *has a matching which covers all but at most* 3 *vertices of* H.

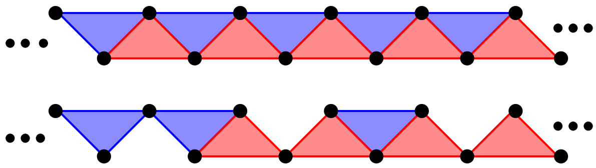
The bound on the minimum degree in Theorem 3 is again best possible. Both Theorem 2 and 3 can be used to prove analoguous results about matchings in 3-uniform hypergraphs which are no longer 3-partite. At the moment, we are trying to eliminate the error term in the bound on the minimum degree in Theorem 2.

Activities

- Colloquium on Combinatorics in Magdeburg (November 2003), talk "Extremal connectivity for topological cliques"
- Habilitation at Hamburg University (November 2003)
- Oberwolfach Conference on Combinatorics (January 2004), talk "Spanning triangulations in graphs"
- talk "Minors in graphs of large girth" in the Colloquium of the Graduate school
- I lectured a course on "Extremal Graph Theory" at FU Berlin

Preview

In 1952 Dirac proved that every graph of order n and minimum degree at least $n/2$ contains a Hamilton cycle. Until recently, only very few results were known about Hamilton cycles in hypergraphs. There are (at least) two ways of defining a Hamilton cycle in a 3-uniform hypergraph – such a cycle can be tied together tightly or loosely:



Very recently Rödl, Ruciński and Szemerédi (in preparation) showed that every 3-uniform hypergraph H of sufficiently large order n which satisfies $\delta_2(H) \geq n/2 + \varepsilon n$ contains a tight Hamilton cycle. Up to the error term εn, the bound on the minimum degree is best possible. Together with Deryk Osthus, I am trying to prove an analogue for loose Hamilton cycles. Here a minimum degree of $n/4 + \varepsilon n$ seems to be sufficent. (If true, this would also be best possible up to the error term εn.) This is work in progress.

References

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