Semester Report WS 03/04 of Stephan Hell

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Field of Research:	Discrete Geometry
Topic:	Topological Methods in Combinatorics and Geometry
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This was my second semester in the group of G. M. Ziegler, and I am continuing my work started in the first semester. The topics *Field of research* and *Recent work and perspectives* of my semester report for the semester SS 03 are still valid. An extensive introduction into the basic ideas and problems can be found there.

Field of Research

The starting point for my work is Jiří Matoušek's book [Mat03]. In this book combinatorial and geometric results, e. g. the Van Kampen–Flores Theorem, are proven using G–index theory where G is a finite group. The main focus is on free $\mathbb{Z}_p$–actions where p is a prime number. For some of the problems, e. g. the Necklace Theorem, it is sufficient to prove the prime number case, then one can apply a direct combinatorial argument for the general case. The problems concerning Tverberg partitions and Kneser colorings don't have this combinatorial structure. Murad Özaydin was the first to prove the prime power case using some deep results from algebraic topology, see [Öz87]. Since then other authors were able to get similar results in various formulations, e. g. A. Yu. Volovikov's lemma on non free, but fixed–point free actions, see [Vol96]. The theory of obstructions and characteristic classes plays a key role in their proofs. In this field it is one of the most interesting questions whether the combinatorial results also hold for general p. One might be able to generalize the arguments from the prime and from the prime power case.

As a side effect I got into topological graph theory. There are relations to graph drawings, crossing numbers and planar graphs.

Recent work and perspectives

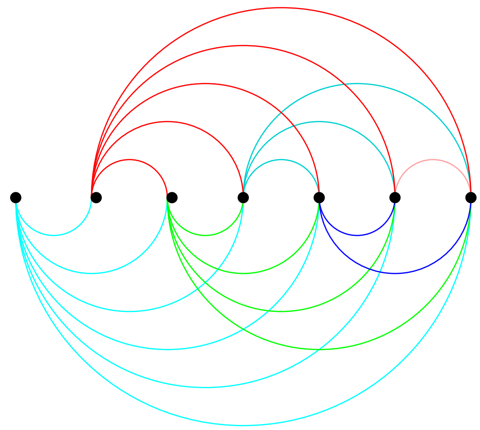
In the following I will present my work which was mainly directed towards the Topological Tverberg Theorem and the Sierksma Conjecture. The first one states the existence of a Tverberg partition $F_1, \ldots, F_q < \Delta^{(d+1)(p-1)}$ for continuous maps $f : \Delta^{(d+1)(p-1)} \to \mathbb{R}^d$, and it is proven for general d and prime powers p. The second one gives the lower bound $((p-1)!)^d$ for the number of Tverberg partitions. To my knowledge it is still open for all d and p – even for affine f – except for the case $d = 1$ which can be proven using the intermediate value theorem and an easy counting argument.

I had regular working sessions with Torsten Schöneborn, diploma student in our group. He had new ideas for the Topological Tverberg Theorem. In his diploma thesis he formulates two related problems: 1) There is a Tverberg partition in the d–skeleton of Δ^N. 2) There is a Winding–number partition in the $(d-1)$–skeleton of Δ^N. Then he shows that both problems are equivalent to the Topological Tverberg Theorem. For $d = 2$ problem 2) leads to the theory of graph drawings: the 1–skeleton of a N–dimensional simplex equals the complete graph K_{N+1}, a map f can then be seen as a map from K_{N+1} to the plane, in other words a drawing of K_{N+1}.

While looking for counter–examples, I tried to establish some relation between drawings having small crossing number and those having small number of Tverberg partitions. In this direction I've checked the drawings that have led to Guy's Conjecture on the crossing number of K_n. I've completely analyzed the family of "bow" drawings; see the figure on page 3 for a bow drawing of K_7. They attain the bound of Sierksma's Conjecture, but their crossing number is $\sum_{k=1}^{\lceil (n-4)/2 \rceil} ((n-4) - 2(k-1))^2$. This family of bow graphs is minimal in second way: Their sets of Tverberg points consist of one point.

Having in mind that the Topological Tverberg Theorem is proven for prime powers, especially for drawings of $K_4, K_7, K_{10}, K_{13}, K_{19}, K_{22}, K_{25}, K_{31}, \ldots$, it is a natural question whether one can fill the gaps by a combinatorial, inductive argument. This problem is still open.

The proof of the prime power case of the Topological Tverberg Theorem by Karanbir Sarkaria can be found in Mark de Longueville's paper [dL99] with a lot of background for the methods from algebraic topology. The initial problem is transformed in several steps. At the end of this process you get a topological problem: Show that some characteristic class – the top chern class – of some specific complex vector bundle does not vanish. In the



second step one attacks this problem; there the fact that p is a prime power is used. The negative result of Theorem 4.2 in [Öz87] seems to prevent a generalization of this method. One can still look at small concrete cases to see what happens.

The software tool *Polymake* developed by Michael Joswig of our group has a new feature called *topaz*. This application is designed to deal with abstract simplicial complexes. One can calculate a wide range of topological invariants such as homology and Stiefel–Whitney classes. It is very useful for checking examples in small dimensions.

In the framework of the seminar *Topologische Kombinatorik* I solved an exercise posed in [Mat03]: Let G be discrete space of cardinality m. Then the space $G^{*(n+1)}$ is known to be homotopy equivalent to a wedge of n–spheres. What is the number of spheres ? The answer is $(m-1)^{(n+1)}$ which can be proven by an induction on m and a second one on n.

In December I have returned to the theory of graph drawings, especially geodesical drawings of K_n. A geodesical drawing of K_n is determined by n points in general position on the sphere S^2. The edges are the geodesics, the locally shortest curves between two vertices. It is known that the drawings of Guy's conjecture have a geodesical representation. I have written a Java program that generates randomly geodesical drawings in a first step. In a second step it calculates their crossing number and number of Tverberg partitions. My test were focused on K_{16}: (i) the lower bound 14400 of Sierksma was attained in 9 instances out approximately 3000, their crossing numbers are far from being minimal. (ii) there were 13 instances having a crossing number in the range of 588 and 602 – the first one is the value from Guy's

Conjecture, 603 is the currently best known rectilinear crossing number from Franz Aurenhammer's website [Aur]. Their numbers of Tverberg partitions are far away from the lower bound.

Currently I'm working towards a new lower bound for the number of Tverberg partitions in the prime power case. A. Vučić and R. Živaljević have proven the lower bound $\frac{1}{(p-1)!}(\frac{p}{2})^{(d+1)(p-1)/2}$ for p prime in 1993, see [Mat03]. One might be able to plug into Volovikov's Lemma instead of the $\mathbb{Z}_p$–index argument. One would get a lower bound for the number of Necklace partitions in the prime power case as a consequence.

Activities

- Lectures and Colloquia of the CGC
- Talks at *Mittagsseminar Diskrete Geometrie* at TU Berlin
- Talk at *Kolkom 2003*, November 14 – 15, Magdeburg
- Talk at seminar *Topologische Kombinatorik* of M. de Longueville at FU Berlin
- Talk at Colloquium of the CGC, February 16, TU Berlin
- Block course *DOCCOURSE BERLIN–PRAGUE 2004*, January 20 – March 5, Prague, as a first part of my long stay

References

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[Öz87] M. Özaydin, *Equivariant maps for the symmetric group*, Preprint, University of Wisconsin–Madison, 1987, 17 pages

[Vol96] A. Yu. Volovikov, *On a topological generalization of the Tverberg theorem*, Math. Notes, 3 (1996), 324–326