Semester Report WS03/04 of Kevin Buchin

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Field of Research:	Computational Geometry
Topic:	Space-Filling Curves
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Field of Research

A *discrete space-filling curve* (DSFC) can be defined as a bijective map from $[n^r]$ to $[n]^r$ where $[m] := \{1, \cdots, m\}$ for $m \in \mathbb{N}$. Of particular interest are DSFC that preserve locality, either in the sense that nearby points in the linear space are mapped to nearby points in the multi-dimensional space, or that the pre-image of nearby points in the multi-dimensional space are nearby in the linear space.

If $A = (a_{ij})_{i,j \in [n^r]}$ measures the nearness or distance in the linear space and $B = (b_{ij})_{i,j \in [n]^r}$ measures respectively the distance or nearness in the multi-dimensional space, then the average locality of a DSFC C can be expressed as

$$\frac{1}{n^r} \sum_{i=1}^{n^r} \sum_{j=1}^{n^r} a_{ij} b_{C(i)C(j)}.$$

By mapping $[n]^r$ back to $[n^r]$ by a fixed bijective map, e. g. $\iota : (k_1, \cdots, k_r) \mapsto \sum_{j=1}^{r} k_j n^{j-1}$ the optimization problem can be formulated as a *Quadratic Assignment Problem*(QAP), i. e.

$$\textbf{Minimize} \ \sum_{i=1}^{n^r} \sum_{j=1}^{n^r} a_{ij} \tilde{b}_{\phi(i)\phi(j)}$$

over all permutations $\phi \in \mathcal{S}_{n^r}$, where $(\tilde{b}_{ij})_{i,j \in [n^r]}$ is defined by $\tilde{b}_{ij} := b_{\iota^{-1}(i) \ \iota^{-1}(j)}$.

Therefore, techniques for QAPs can be used to optimize discrete space-filling curves. To two cases in two dimensions, I applied QAP heuristics to two cases in two dimensions: the *average path-length* Δ_q between neighboring points weighted by the exponent q and the *expected tour-length* $E(L)$ of a Probabilistic Traveling Salesman (PTSP) on a grid. For Δ_q the matrices are given by $a_{ij} = |i - j|^q$ and B is the adjacency matrix of the grid points. For $E(L)$ we have $a_{ij} = p_{i \to j}$, where $p_{i \to j}$ denotes the probability that the point with the index i is visited and the point with the index j is visited next, and B is the distance matrix of the grid points.

The heuristics applied were simulated annealing, taboo search and an ant algorithm. If the points in the PTSP are to be visited with a small probability, the best DSFC for $E(L)$ calculated by the heuristics traverses the points in a nearly angular ordering, i. e. an ordering of the points by their angle, with the centre of the grid as origin. In the following, I looked at the Probabilistic Traveling Salesman Problem in more detail.

One approach taken, is to bound $E(L)$ using eigenvalue based bounds for the QAP. For $E(L)$, the coefficient matrices A, B are symmetric matrices with real entries. Let $\lambda = (\lambda_1, \cdots \lambda_n)$ denote the eigenvalues of A and $\mu = (\mu_1, \cdots, \mu_n)$ denote the eigenvalues of B. The simplest eigenvalue based bounds for the assignment problem are $\langle \lambda, \mu \rangle^-$ as lower bound and $\langle \lambda, \mu \rangle^+$ as upper bound, where $\langle \lambda, \mu \rangle^-$ and $\langle \lambda, \mu \rangle^+$ denote respectively the smallest and largest possible value of the scalar product, allowing permutation of the coordinates. But so far the bounds achieved are too weak to be useful.

In the probalistic analysis of the traveling salesman problem, (non-discrete) SFC are used to prove concentration bounds for the length of the tour using Talagrand's Inequality [1]. The techniques used also apply to other optimizations problems in the Euclidean plane, like the minimal edge length of a minimal spanning tree of random points. Therefore, I studied the probability theory of Euclidean optimization problems.

Activities

Talks

- *Locality properties of discrete space-filling curves*, 3rd Workshop on Combinatorics, Geometry, and Computation, in Neustrelitz
- *Optimal Numberings of a Square Grid*, Noon Seminar of the TI-AG, FU Berlin
- *Concentration of Measure using Talagrand's Inequality*, Noon Seminar of the TI-AG, FU Berlin

Attended events

- *CGC monday lectures and colloquia*

- Lecture on *Selected Topics in Computational Geometry* by Dr. Christian Knauer at the FU Berlin
- Lecture on *Convex Geometry* by Dr. Ivan Izmestiev at the FU Berlin
- *Noon Seminar* of the TI-AG at the FU Berlin
- *4th Max-Planck Advanced Course on the Foundations of Computer Science* (ADFOCS 2003) at the MPI in Saarbrücken
- 3rd Workshop on *Combinatorics, Geometry, and Computation*, in Neustrelitz
- Fall School on *Computational Geometry*, in Neustrelitz

Other activities

- Research visit with Dr. M. Costa Sousa at the University of Calgary
- Referee for SODA 2003
- Referee for the International Journal of Computational Geometry and Applications

Preview

January 26 - March 2, 2004, Doccourse at the Charles University in Prague

References

[1] Steele, J. M. *Probability Theory and Combinatorial Optimization*, SIAM CBMS **69**, 1997.