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Field of Research:	Discrete and Combinatorial Geometry
Topic	Topology in DCG
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Field of Research

This semester I continued my work in the field of topology and its applications to graph theory. In joint work with Péter Csorba, Carsten Lange and Ingo Schurr we improved a result of us and showed that graphs without a bipartite $K_{k,l}$-subgraph have a box complex of small index.

In 1978 Lovász [2] proved Kneser's conjecture, a theorem of graph theory, using a classical topological ingredient, the Borsuk-Ulam theorem. Kneser's conjecture is concerned with the chromatic number of certain graphs. The Borsuk-Ulam theorem asserts that there is no continuous antipodal map from a sphere to its equator, i. e. $\nexists f : S^{d+1} \to S^d, f(-x) = -f(x)$,

Over the years Lovász' idea became the following (see [3]). Given a graph G, use all its complete bipartite subgraphs to define a simplicial complex $\mathcal{B}(G)$. In this so called *box complex* $\mathcal{B}(G)$ every simplex has a unique antipodal counterpart, i. e. there is a free $\mathbb{Z}_2$-action on $\mathcal{B}(G)$. On the sphere S^d we also have a $\mathbb{Z}_2$-action by the antipodal map. Hence we can define the *index* of the complex $\mathcal{B}(G)$ as the lowest dimension d of a sphere S^d into which we can map $\mathcal{B}(G)$ $\mathbb{Z}_2$-equivariantly:

$$\mathrm{ind}(\mathcal{B}(G)) := \min\{d \mid \exists f : \mathcal{B}(G) \xrightarrow{\mathbb{Z}_2} S^d\}$$

Based on the Borsuk-Ulam theorem Lovász proved the following essential relation between the chromatic number of a graph and the index of this complex (see [4]).

$$\chi(G) \geq \mathrm{ind}(\mathcal{B}(G)) + 2.$$

This formula gives us a lower bound for the chromatic number of a graph. But this lower bound is not always tight. In joint work with Péter Csorba, Carsten Lange and Ingo Schurr we proved an upper bound to this lower bound. If a graph G does not contain a complete bipartite graph $K_{k,l}$ as a subgraph then the index of its box complex is bounded by $\mathrm{ind}(\mathcal{B}(G)) \leq k + l - 3$.

To show this we constructed a $\mathbb{Z}_2$-equivariant map from the box complex $\mathcal{B}(G)$ to a simplicial complex of dimension at most $l + k - 3$. Since the dimension of a space is an upper bound to its index this proves the assertion.

A negative application of this result is the chromatic number of the plane $\mathbb{R}^2$. The nodes of the corresponding graph are all points in the plane. Two of them are connected by an edge if their Euclidean distance is 1. This graph does not contain a $K_{2,3}$-subgraph since two circles intersect in at most two points. Hence the corresponding index is at most 2. This shows that Lovász' bound could yield at most the well known bound $\chi(\mathbb{R}^2) \geq 4$ but cannot prove $\chi(\mathbb{R}^2) \geq 5$.

Another project, with Volker Kaibel, was an article about the combinatorial symmetries of cyclic polytopes. Cyclic polytopes are extremal: Among all polytopes of dimension d with n vertices the cyclic polytopes $C_d(n)$ have in each dimension the maximum number of faces. (Upper bound theorem, see [5]) Moreover their combinatoial structure is very easy to describe (Gale's evenness criterion, see [1]). It is probably well-known to most polytope theorists that the group of combinatorial symmetries of cyclic polytopes $C_d(n)$ is isomorphic to the dihedral group of order n for even d and to $\mathbb{Z}_2 \times \mathbb{Z}_2$ for odd d — if $n \geq d + 3$. However, it seems that this is not accessible at any prominent place in the literature. Our detailed elaboration may not only serve as an example for the determination of combinatorial automorphism groups of polytopes, but it also reveals the variational structures for $n = d + 2$. Moreover, we showed that each cyclic polytope can be realized in Euclidean space such that all its combinatorial symmetries are induced by self-congruences.

Since January 2003 I am spending my "long-stay" at Emo Welz's group at ETH Zürich, Institute of Theoretical Computer Science. I attended the block courses by Markus Gross on surface representations and geometric modeling and by József Beck on positional games. Now I am curious about what will happen in the next months.

Publications

- *Automorphism Groups of Cyclic Polytopes* (with Volker Kaibel) to appear as Chapter 8 of *Triangulated Manifolds with Few Vertices* by Frank H. Lutz.

Activities

- Presentation of the talk *A Quantified Version of the Borsuk-Ulam Theorem* at the *2nd Workshop on Combinatorics, Geometry, and Computation,* Hiddensee, October 9 – 12, 2002.
- Presentation of the talk *Bounds of the Topological Method* at *Kolloquium über Kombinatorik,* Magdeburg, November 15 – 16, 2002.
- Presentation of the talk *Die Symmetrien der zyklischen Polytope,* Oberseminar, TU Berlin, December 13, 2002.
- Presentation of the talk *Shellable and Cohen Macaulay partially ordered sets,* December 18, 2002, at the seminar *Topologische Kombinatorik* by Mark de Longueville, FU Berlin.
- Attendance of the lecture *Algorithmische Graphentheorie* by Ekkehard Köhler, TU Berlin.
- Attendance of the block courses *Surface Representations and Geometric Modeling* by Markus Gross and *Positional Games* by József Beck, January 6 – February 7, 2003, ETH Zürich.

References

[1] David Gale, *Neighborly and cyclic polytopes*, in Proc. Sympos. Pure Math., vol. VII, Providence, R.I., 1963, Amer. Math. Soc., pp. 225 – 232.

[2] László Lovász, *Kneser's conjecture, chromatic number, and homotopy*, Journal of Combinatorial Theory, Series A, **25** (1978), pp. 319–324.

[3] Jiří Matoušek and Günter M. Ziegler, *Topological lower bounds for the chromatic number: A hierarchy.* preprint, August 2002.

[4] Jiří Matoušek with the collaboration of Anders Björner and Günter M. Ziegler, *Using the Borsuk-Ulam theorem.* Book to be published by Springer.

[5] Günter M. Ziegler, *Lectures on Polytopes*, no. 152 in Graduate Texts in Mathematics, Springer-Verlag, Berlin, 1995.