Semester Report WS02/03 of Shi Lingsheng

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Topic:	Ramsey Theory
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Field of Research

The Ramsey number $R(G, H)$ of two simple graphs G and H is the least integer r so that for any bicoloring of the edges of a complete graph of order r there is either a monochromatic copy of G or a monochromatic copy of H. By a (special case of) well known theorem of Ramsey [13], this number is finite for all graphs G and H. The determination or estimation of these numbers is usually a very difficult problem. For classic Ramsey numbers of complete graphs, the only values that are known precisely are those of $R(K_3, K_n)$ for $n \leq 9$, $R(K_4, K_4)$ and $R(K_4, K_5)$. Even the determination of the asymptotic behavior of Ramsey numbers for a set of graphs up to a constant factor is a hard problem, and despite a lot of efforts by various researchers (see, eg. [5, 7] and their references), there are only a few infinite set of graphs for which this behavior is known. A particular interesting example of an infinite set for which the asymptotic behavior of the Ramsey numbers is known, is the result of Kim [9] together with that of Ajtai, Komlós and Szemerédi [1]: $R(K_3, K_n) = \Theta(n^2/\ln n)$.

Results

We say that the Ramsey numbers *grow linearly* for a set of graphs $\mathcal{G}$ versus a set of graphs $\mathcal{H}$ if there is a constant $c = c(\mathcal{G}, \mathcal{H}) \geq 1$ so that $R(G, H) \leq cn$ for all $G \in \mathcal{G}$ of order n and $H \in \mathcal{H}$ of order n. In particular, we simply say that the Ramsey numbers grow linearly for the set of graphs $\mathcal{G}$ if $\mathcal{G} = \mathcal{H}$. It is known for a long time that $R(G, G)$ is exponential in the order of dense graphs G. For example, $2^{n/2} \leq R(K_n, K_n) < 2^{2n}$. Meanwhile, it is also known that the Ramsey numbers grow linearly for very sparse graphs. For example, they grow linearly for paths, cycles, stars and trees. A graph is *d-degenerate* if its subgraphs all have minimum degree at most d. In 1973, Burr and Erdős [3] offered a total of \$25 for settling the following conjecture.

Conjecture *The Ramsey numbers grow linearly for d-degenerate graphs.*

But they also wrote in [3] "However, it seems to be quite difficult, and probably further work must continue to be in the direction of partial results". In fact, some weakened versions of this conjecture were obtained in the last two decades. In 1983, Chvátal, Rödl, Szemerédi, and Trotter [6] proved that the Ramsey numbers grow linearly for all graphs with bounded maximum degree. In 1993, Chen and Schelp [4] extended this to *p-arrangeable* graphs which are those whose vertices can be ordered as $v_1, v_2, ..., v_n$ so that for each integer i with $1 \leq i \leq n$, at most p vertices among $\{v_1, v_2, ..., v_i\}$ have a neighbor $v \in \{v_{i+1}, v_{i+2}, ..., v_n\}$ adjacent to v_i. They also showed that a planar graph is 761-arrangeable, which was later improved to 10-arrangeable by Kierstead and Trotter [8]. Thus their results imply that the Ramsey numbers grow linearly for all planar graphs. In 1997, Rödl and Thomas [14] extended this to graphs with bounded genus and proved that all graphs without any topological minor of a clique of order p are p^8-arrangeable and so make their Ramsey numbers linearly grow. As pointed out in [4], subdivided graphs need not be p-arrangeable. But in 1994, Alon [2] proved that the Ramsey numbers still grow linearly for them.

In 2001, Kostochka and Rödl [10] extended Alon's result to crowns (a special kind of bipartite sparse graphs), which confirms a conjecture by Burr and Erdös [3] as well as by Trotter [10]. In [15], we extend the result of Kostochka and Rödl to the statement that the Ramsey numbers grow linearly for degenerate graphs versus crowns.

Recently, Kostochka and Rödl [11] also proved that the Ramsey number $R(G, H)$ of a d-degenerate graph G of order n and a d-degenerate graph H of order n with maximum degree Δ is bounded by $cn\Delta$. If Δ is bounded, this implies that the Ramsey numbers grow linearly for degenerate graphs versus graphs with bounded maximum degree. We extend this and the result of Chen and Schelp to degenerate graphs versus p-arrangeable graphs. Now by combining the previous results, we know that the Ramsey numbers grow linearly for degenerate graphs versus planar graphs or graphs without any topological minor of a fixed clique. Recently, Kostochka and Sudakov [12] obtained the almost linear bound $R(G, G) < n^{1+o(1)}$ for all d-degenerate graphs G of order n. However, the conjecture of Burr and Erdős is still open despite all the efforts up to now.

Activities

- Satellite Conference of ICM-2002 "Combinatorics, Graph Theory and Applications" with a talk on "Polynomial Ramsey numbers for bipartite O(ln$n$)-degenerate graphs of order n"), Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China, August 15 - 17, 2002
- International Congress of Mathematicians, Beijing, China, August 20 - 28, 2002
- Combinatorics in honour of Walter Deuber's 60th Birthday, Humboldt-Universität zu Berlin, Germany, October 7 - 8, 2002
- Annual Workshop on Combinatorics, Geometry, and Computation (with a talk on "Ramsey numbers of sparse graphs"), in Hiddensee, Germany, October 9 - 12, 2002
- "Learn&work"-shop in Berlin, Germany, November 7 - 9, 2002
- 30. Berliner Algorithmen Tag, Konrad-Zuse-Zentrum für Informationstechnik, Berlin, February 7, 2003
- Lectures and colloquia of the European graduate program "Combinatorics, Geometry, and Computation" (with a talk on "Do Ramsey numbers grow polynomially for O(ln$n$)-degenerate graphs of order n?"), Winter Term, 2002
- Seminar "Algorithms and Complexities" at the HU (with a talk on "Piecewise syndetic sets"), Winter Term, 2002

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