Semester Report WS 02/03

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Field of Research:	Discrete Geometry
Topic:	Flag Vectors of Polytopes
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Field of Research and Results

In this semester I continued my work on the construction and realization of 2-simple and 2-simplicial 4-polytopes. In my last report I already discussed the relation of this question to the general problem of proving tight bounds on the entries of the flag vector for 4-dimensional polytopes. It was noted by Bayer [Bay87] that our knowledge of non-simple and non-simplicial 4-polytopes is incomplete and that for an improvement of the flag vector inequalities an infinite family of 2-simplicial and 2-simple ($2s2s$) 4-polytopes would be helpful. Such a family was presented by Eppstein, Kuperberg, and Ziegler in [EKZ02], but the construction is quite complicated, applies only to a very restricted class of polytopes and a priori produces polytopes with non-rational coordinates. The actual state of affairs can be found in Ziegler's survey [Zie02] for the ICM.

The restriction on the polytopes for the E-construction of Eppstein, Kuperberg, and Ziegler is that they need simplicial 4-polytopes that have their edges tangent to the unit sphere $\mathbb{S}^3$. In dimensions 4 and higher this is difficult to satisfy. Already in the last semester I realized that – though it is quite helpful for the proof – this edge-tangency requirement is not neccessary for the construction to work. By direct computation I managed to give rational representations for many of the polytopes of [EKZ02] and construct a simple new class which cannot be dealt with by the original construction. But the resulting construction method was complicated and unsatisfying.

In this semester in a joint effort with my supervisor we managed to find a very clear and easy construction that applies to Eulerian lattices of any rank r which, when applied to the face lattice of a polytope, contains all the examples above. We will present this construction in [PZ03]. We call our construction the "E_t-construction", where t is a parameter between 0 and $d-1=r=2$ (d is the dimension in the case of a face lattice of a polytope). Given an Eulerian lattice L, we construct a new lattice in the following way:

$E_t(L)$ is the lattice that consists of the following subsets of L, ordered by reversed inclusion:

- the empty set,
- the one element sets $\{y\}$ for $y \in L_t$, and
- the intervals $[x, z] \subseteq L$ such that some $y \in L_t$ satisfies $x < y < z$.

We can show that this lattice is again Eulerian. In case we start with the face lattice of a piecewise linear sphere we show that the resulting lattice $E_t(L)$ is again the lattice of a piecewise linear sphere. Further we examine under which conditions on the lattice or sphere the result will be 2-simple and k-simplicial for some k between 2 and $d-2$. Unfortunately it is easy to see that we can obtain h-simple lattices for $h > 2$ by this construction only in the trivial cases $t = d-1$ and $t = 0$ when we get back the lattice or its opposite.

This part of the construction is very clear and insightful. The problems start with the next step when we want to apply the construction to face lattices of polytopes and determine when we can construct a polytope with that face lattice. We have no general criterion which tells us whether we can realize $E_t(P)$ as a polytope or not. But we can present some quite general classes of polytopes where we can explicitly give a realization. First, our construction applies to any polytope to which the original E-construction of Eppstein, Kuperberg, and Ziegler applied. It also contains all examples of 2-simple and $(d-2)$-simplicial d-polytopes that were mentioned in the book of Grünbaum [Grü67]. The interesting case, that is the one giving $2s2s$ polytopes, is $t = 1$ for 4-polytopes. By a "Vertex Truncation" method we can e.g. show that we can (rationally) realize $E_1(P)$ for any stacked polytope P. This proves that there are infinitely many rational $2s2s$ polytopes. By a generalization of the proofs in [EKZ02] we can show that there are infinitely many 2-simple and $(d-2)$-simplicial d-polytopes in any dimension $d \geq 3$. A similar construction, but only for regular polytopes, can be found in the book of Coxeter [Cox63]. However, he has a completely different aim, namely the construction of further regular and semi-regular polytopes.

Still this construction leaves some open question which we want to deal with in the future. There are many polytopes for which we are not able to say wether we can realize $E_t(P)$ for any t as a polytope or not. Also, there are many polytopes for which we know that the construction applies and gives a polytope, but we can only write down a representation with non-rational coordinates.

In the last weeks I took up a problem I also considered in the last semester. I worked on the problem of either bounding the fatness of polytopes above or constructing polytopes of arbitrary high fatness. Here "fatness" is the quotient of the sum of the numbers of edges and ridges versus the sum of vertices and facets. This parameter was introduced in [EKZ02] to aid the classification of flag vectors. On the other hand I considered the question whether fatness is bounded below for cellular spheres, as we know a lower bound of the fatness of polytopes by Kalai's lower bound theorem but no such bound for (non-simplicial) spheres.

Activities

- Talk on "Some New Constructions for Polytopes" at the Colloquium of the CGC, TU Berlin, January 13, 2003
- CGC Annual Workshop, Hiddensee, October 9 –12, 2002
- Conference "Diskrete Mathematik", TU Dresden, October 3 –4, 2002
- Conference "Prospects in Geometry", MPI Leipzig, November 28 – 29, 2002
- Conference in Honour of Jürgen Bokowskis 60^{th} Birthday, TU Darmstadt, February 2 – 4, 2003
- Berliner Algorithmentag February 7, 2003, at ZIB Berlin
- Attended the Lectures and Colloquia of the CGC
- Attended the Lecture on "Computational Geometry" by Michael Joswig
- Attended the Oberseminar and the Brown-Bag-Seminar at TU Berlin

  Talk in the Oberseminar on "Constructions for Polytopes", November 27, 2002

Preview

In the next few weeks I hope to finish the work on the E_t-construction and the construction of 2-simple and $(d-2)$-simplicial d-polytopes. In the following time I want to pursue two different directions. As already mentioned, though we have a nice construction for infinitely many 2-simple and 2-simplicial 4-polytopes, there remain many open questions connected with this construction. But secondly I want to work on the problem of finding a statement similar to the lower bound theorem for polytopes and simplicial spheres in the case of general cellular spheres. This would be an important step towards my general goal of improving the known facts about the classification of f-vectors and flag vectors of 4-polytopes.

References

[Bay87] MARGARET M. BAYER, *The extended f-vectors of 4-polytopes*, J. Combin. Theory Ser. A, 1987, vol. 44, pp. 141–151.

[Cox63] H. S. M. COXETER, *Regular Polytopes*, Macmillan, New York, second ed., 1963. Corrected reprint, Dover, New York 1973.

[EKZ02] D. EPPSTEIN, G. KUPERBERG, AND G. M. ZIEGLER, *Fat 4-polytopes and fatter 3-spheres*, in Discrete Geometry: In honor of W. Kuperberg's 60th birthday, A. Bezdek, ed., Pure and Applied Mathematics. A series of Monographs and Textbooks, Marcel Dekker Inc., 2003, pp. 239–265. `arXiv:math.CO/0204007`.

[Grü67] B. GRÜNBAUM, *Convex Polytopes*, vol. 221 of Graduate Texts in Math., Springer-Verlag, New York, 2003, in print. Second edition edited by V. Kaibel, V. Klee and G. M. Ziegler (original edition: Interscience, London 1967).

[PZ03] A. PAFFENHOLZ AND G. M. ZIEGLER *The E_t-construction for lattices, spheres, and polytopes*, in preparation

[Zie02] G. M. ZIEGLER, *Face numbers of 4-polytopes and 3-spheres*, in Proceedings of the International Congress of Mathematicians, (ICM 2002, Beijing), L. Tatsien, ed., vol. III, Beijing, China, 2002, Higher Education Press, pp. 625–634. `arXiv:math.MG/0208073`.