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Field of Research:	Discrete and Combinatorial Geometry
Topic	f-vectors of Polytopes
PhD Student	since June 1, 2001

Field of Research

Given a d-dimensional polytope its f-vector $(f_0, f_1, \dots f_{d-1})$ denotes its number of vertices, edges, ..., and facets. It is an old problem to determine the set of all f-vectors of polytopes. In dimension 3 the answer is known since 1906. Steinitz' theorem asserts that the set of all f-vectors of 3-polytopes is the set $\{(f_0, f_1, f_2) \in \mathbb{Z}^3 | \; f_1 = f_0 + f_2 - 2, \; 2f_2 - 4 \geq f_0 \geq 4, \; 2f_0 - 4 \geq f_2 \geq 4\}$. These points are the integer vectors of a 2-dimensional cone in 3-space.

In dimension 4 the problem is open. Six linear inequalities (e.g. $2f_1 \geq f_0$) and four non-linear inequalities (e.g. $f_1 \leq \binom{f_0}{2}$) are known. But the set of f-vectors has a more complicated shape than in dimension 3. It is not a cone. Moreover it is not "convex" any more, i.e. the convex hull of the set of f-vectors contains integer vectors that are not f-vectors of any 4-polytope. Moreover this convex hull is not a closed set [1]. Although we do not know the shape of the entire set of 4-dimensional f-vectors its projections to the coordinate planes are known. For every pair of components (e.g. (f_0, f_2)) we know what combinations can occur (see [2]).

f-vectors also can be defined for strongly regular CW-decompositions of spheres. A CW-decomposition is strongly regular if the boundary of every cell is embedded and the intersection of any two closed cells is a closed cell. Steinitz' theorem implies that every s.r. cell decomposition of the 2-sphere is combinatorially equivalent to the boundary of a 3-polytope. In this sense 2-spheres and 3-polytopes are "the same." In particular the respective sets of f-vectors are the same.

One dimension higher the situation is different. There are s.r. cell decompositions of the 3-sphere which are not combinatorially equivalent to the boundary of any 4-polytope. It is an open question if the respective sets of f-vectors are different. A possible candidate to distinguish these two sets of f-vectors is a parameter called *fatness* which is defined to be the quotient $\frac{f_1+f_2}{f_0+f_3}$. Using hyperbolic surfaces Eppstein, Kuperberg, and Ziegler [3] have constructed an infinite family of 3-spheres with arbitrary high fatness. But the known examples of fat 4-polytopes have fatness at most 5.06. Thus it is conjectured that fatness of polytopes is bounded.

One way to investigate the fatness of a 4-polytope is to construct a tiling from it. To do this one first tiles Euclidean 3-space by congruent simplices. Then one constructs a Schlegel diagram over a simplex facet. (See [5] for a definition of a Schlegel diagram.) The last step is to insert into each simplex of the tiling one copy of the Schlegel diagram. This yields the desired 3-dimensional tiling.

But instead of 3-dimensional tilings I first considered 2-dimensional tilings. The analogue to fatness in this 2-dimensional case is a tiling with tiles that have many vertices. I considered shellings of fat tilings and defined a parameter that grows with each tile. The parameter is chosen such that it grows the quicker the fatter the tiling is. If the tiling is *normal* (see [4, sect. 3.3] for a definition) a high growth rate of this parameter implies that the tiling "needs much area." In hyperbolic space such fast growing tilings can be realized. In Euclidean space this is not always possible. Using this growing parameter I could prove that normal tilings in Euclidean plane cannot have more vertices per tile than the hexagonal tiling has.

Another way to prove this result is to define an analogue to an f-vector for tilings. Its components are the average numbers of vertices per tile, edges per tile, etc. With the notion of an f-vector of a tiling one can prove analogues to Euler's formula. One example can be found in Grünbaum & Shepard [4, sect. 3.3]. Unfortunately this version cannot be generalized in higher dimensions. But with a shorter proof I found a slightly different version of Euler's formula that easily can be generalized to higher dimensions.

Activities

- 27. Berliner Algorithmentag (July 6, 2001)
- *15. ÖMG-Kongress, Jahrestagung der Deutschen Mathematikervereinigung* (Vienna, September 16 - 22, 2001)
- CGC Fall School *Discrete Geometry - Triangulations from various points of view* (Alt-Ruppin, October 4 - 6, 2001)
- Block courses of CDC's Pre-Doc program. *Randomized Algorithms* by Emo Welzl and *Topological Methods in Combinatorics and Geometry* by Jiri Matousek (Zürich, October 22 - November 23, 2001)

- *Mittagsseminar* of Emo Welzl's research group *Theory of Combinatorial Algorithms* during my stay at ETH Zürich, including a talk on *Cyclic Polytopes and Their Symmetries* (November 6, 2001)

- Lectures and colloquia of the CGC, including a talk in the colloquium on *f-Vectors of Polytopes and Tilings* (February 4, 2002)

- Lecture *ADM II, Lineare Optimierung* by Günter M. Ziegler at TU Berlin.

- *Brown Bag Seminar* about differential and discrete geometry at TU Berlin, including talk on *f-Vektoren von Polytopen und Pflasterungen* (January 29, 2002)

- *Oberseminar Diskrete Geometrie* at TU Berlin

- *Reading Seminar* B. Grünbaum *Convex Polytopes* at TU Berlin

References

[1] Margaret Bayer, *The extended f-vectors of 4-polytopes*, J. Combinatorial Theory, Ser. A, **44** (1987), pp. 141–151.

[2] Margaret M. Bayer and Carl W. Lee, *Combinatorial aspects of convex polytopes*, North-Holland, 1993, ch. 2.3, pp. 485–534.

[3] David Eppstein, Greg Kuperberg, and Günter M. Ziegler, *Fat 4-polytopes and fatter 3-spheres.* preprint, 20 pages.

[4] Branko Grünbaum and Geoffrey Colin Shephard, *Tilings and Patterns*, W. H. Freeman, 1987.

[5] Günter M. Ziegler, *Lectures on Polytopes*, no. 152 in Graduate Texts in Mathematics, Springer-Verlag, Berlin, 1995.