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Field of Research:	Discrete and Combinatorial Geometry
Topic:	Triangulated Complexes
PhD Student	at the program since March 1, 2000

Field of Research and Results

In this semester, I continued my work on bounding the *maximum* number of pivot steps that the simplex algorithm can take on simple d-polytopes when using the worst possible pivot rule. This approach is in some sense complementary to the famous *Hirsch Conjecture*, which asks if there always exists a *short* path (for example, of length polynomial in the dimension and the number of facets) from source to sink in any linear orientation of a simple d-polytope. The outcome most favorable to linear programming practice would be that (a) yes, there is always a short path (so that we stand a chance of designing a pivot rule that finds it), and (b) no, there never is a really long path (so that it doesn't matter too much whether our pivot rule is bad). Unfortunately (for the practitioners), the Hirsch Conjecture remains open despite decades of effort, and statement (b) is toppling, as I now outline.

I first concentrated on classifying with respect to realizability the candidate orientations of some simple 4-polytopes with few facets, but maximal number of vertices. Here I built on work by CLAUDIA SCHULTZ [**?**], who combinatorially enumerated all orientations of the graphs of these polytopes that satisfy three necessary conditions for realizability of a longest path: They must *admit* a Hamiltonian path; all faces must have a unique sink; and in each k-dimensional face, $2 \le k \le d(= 4)$, there have to exist k edge-independent paths from source to sink (this is the *Holt-Klee condition* [**?**]). For the unique simple 4-polytope with 7 facets and maximal number of vertices (14), I was able to prove that of the 7 equivalence classes of candidate orientations of its graph, exactly 4 are realizable, and the other 3 are not. This was made possible using the *Gale*$^{\Delta}$*-transform* I had learned during my one-semester stay in Zurich.

Perhaps more surprisingly, I was later even able to find realizations of simple 4-polytopes with 8 facets and maximal possible number of vertices (20) that admit a strictly ascending Hamiltonian path. Some of the surprise stems from the fact that by the results of CLAUDIA SCHULTZ's enumeration, the "usual suspects", i.e. the polars of cyclic 4-polytopes with 8 or 9 vertices, do *not*, even combinatorially, admit candidate orientations. However, I found that *both* of the other two combinatorial equivalence classes of simple 4-polytopes with 8 facets and maximal possible number of vertices that exist besides $C_4(8)^{\Delta}$ can be realized in such a way.

Quite recently, I was able to generalize these results and construct, for every integer $n \geq 5$, a realization of a simple 4-polytope with n facets and maximally many vertices that admits a strictly ascending Hamiltonian path with respect to a linear objective function! This means that at least in dimension $d = 4$, the statement (b) above is false. The essential ingredient in the inductive construction is the operation of *facet splitting* introduced by David Barnette [**?**].

Activities

- Attendance of the lectures and colloquia of the CGC
- Presentation of the talk *Kalai's Squeezed 3-Spheres are Polytopal* at the *EuroConference on Combinatorics, Graph Theory and Applications*, September 12–15, 2001, in Barcelona
- Presentation of the talk *Long ascending paths on small polytopes* at the CGC colloquium, November 26, 2001, at FU Berlin
- Presentation of the talk *Der Satz von Steinitz: konstruktiv (und quantitativ)* at the *Brown Bag Seminar*, December 18, 2001, at TU Berlin
- CGC Fall School *Discrete Geometry - Triangulations from various points of view*, October 4–6, 2001, in Alt-Ruppin, Brandenburg
- *15. ÖMG-Kongress, Jahrestagung der Deutschen Mathematikervereinigung,* September 16–22, 2001, in Vienna
- Attended the lectures *ADM II: Lineare Optimierung* (Ziegler) and *Random Walk Methoden in der Kombinatorik* (Ziegler, Kaibel), and the seminar *Algorithmisches Zählen* (Ziegler, Kaibel) at TU Berlin.
- Attended the seminars *Oberseminar Diskrete Geometrie*, *Reading Seminar* B. Grünbaum: *Convex Polytopes*, and *Brown Bag Seminar* at TU Berlin

Publications

- *Kalai's squeezed 3-spheres are polytopal*, revised version (October 2001), 11 pages, `math.CO/0110240`. Accepted for publication in *Discrete & Computational Geometry*

- *Computing Triangulations Using Oriented Matroids*, joint work with Jörg Rambau, 26 pages (December 2001), submitted for publication

Preview

Is $M(d,n) = M_{\mathrm{ubt}}(d,n)$, for $n > d \geq 4$? In other words, is the maximal number of vertices in a strictly ascending path in the graph of a simple d-polytope with n facets equal to the maximal number of vertices that such a polytope can have according to the Upper Bound Theorem [**?**]?

References

[1] David Barnette, *A family of neighborly polytopes*, Israel J. of Math. **39** Nos. 1–2 (1981) 127–141

[2] Fred Holt and Victor Klee, *A proof of the strict monotone 4-step conjecture*, Contemp. Math. **223** (1999) 201–216

[3] Peter McMullen, *The number of faces of simplicial polytopes*, Israel J. of Math. **9** (1971) 559-570

[4] Claudia Schultz, *Schwierige lineare Programme für den Simplex-Algorithmus*, Diplomarbeit, TU Berlin, 2001, 81 pages