Semester Report SS 05 of Arnold Waßmer

Name:	Arnold Waßmer
Supervisor:	Prof. Dr. Günter M. Ziegler
Field of Research:	Discrete and Combinatorial Geometry
Topic:	Topology in DCG
PhD Student	at the program since June 1, 2001
Associated member	

Thank you!

My thesis is finished and submitted. Since this is my last semester report I want to thank all members of the graduate program who made this work possible: the coordinators Bettina Felsner and Andrea Hoffkamp, and the speaker of the program Helmut Alt. Especially I want to thank my supervisor Günter M. Ziegler for his initial inspiration for this project and his enduring support. Thank you very much!

Field of Research and Results

My thesis combines two basic mathematical concepts: the concept of duality and the concept of independent sets of graphs. Duality has several meanings in mathematics. We use it in the sense of dual polytopes. The octahedron and the cube, for example, are dual. The octahedron has 8 triangles, 12 edges, and 6 vertices. These faces form a partially ordered set (poset) – they are ordered by inclusion. Each vertex is contained in four edges; each edge is contained in two triangles. Now we reverse this partial order, which yields the so called dual poset. We ask an innocent question: "Is there a polytope that corresponds to this dual poset?" The answer is "Yes, the cube." The cube has 8 vertices, 12 edges, and 6 squares. Each square contains four edges, each edge contains two vertices. Thus each vertex of the octahedron corresponds to a square of the cube and each triangle of the octahedron corresponds to a vertex of the cube. We say the octahedron and the cube have dual face posets and they are dual polytopes. In general every polytope has a dual polytope; see [20].

The second concept is that of independent sets in graphs. A graph G is a set of nodes, some of which are connected by edges. A subset of the set of nodes is independent if it does not contain any pair of nodes that are connected. For example an independent set of the graph shown in Figure 2 is $\{1, 3, 4, 6\}$. Now we define a geometric object from these independent sets, the independence complex

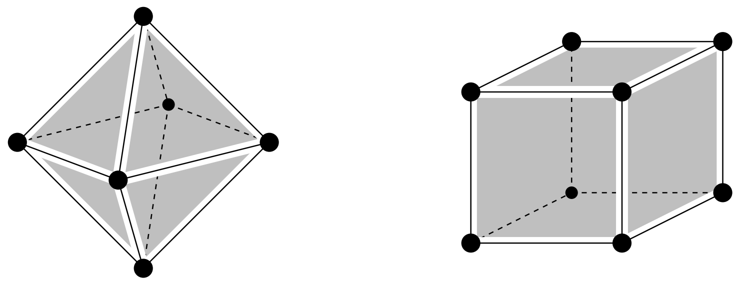

Figure 1: The octahedron and the cube are dual.

IC(G). It is a simplicial complex, which means that it consists of simplices. The simplices of dimensions $0, 1, 2$ and 3 are vertices, edges, triangles and tetrahedra. The simplices of the independence complex IC(G) are the independent sets of the graph G. Thus each vertex of IC(G) corresponds to a node of the graph G. Two nodes that are not connected in the graph now define an edge in the complex IC(G). Three nodes that are independent form a triangle in IC(G). An independent set with four nodes defines a tetrahedron and so on. For an example see Figure 2.

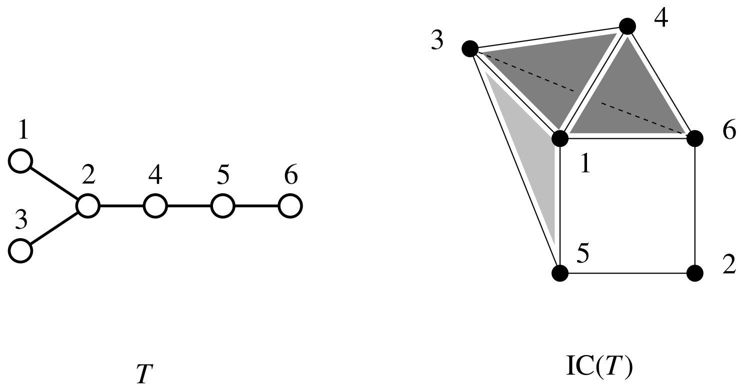

Figure 2: A graph T and its independence complex IC(T).

Independent sets form an important model structure for optimization. An example is coding theory. To design a code means to try to find a maximum set of words of a given length of a given alphabet such that any two words of this set are "very different", which is needed for error correcting. The Hamming graph represents this information. Its nodes are the words of the alphabet of given length. They are connected by an edge if they are not "very different." A code of maximal size now is equivalent to an independent set of maximal size in the Hamming graph. Neil J. A. Sloane maintains a list [18] of graphs arising in this way from coding theory. Finding independent sets of maximal size in general graphs is NP-hard [12]. This is also hard from a practical point of view – in contrast to several other problems, for example the traveling salesman problem, for which one can solve non-trivial instances that have about 25'000 nodes; see [1], [2].

Independent sets are even more striking in another class of hard problems: graph coloring. A coloring of a graph G assigns a color to each node such that neighbors get different colors. Thus nodes that get the same color form an independent set. The chromatic number $\chi(G)$ is the minimum number of colors that is needed for coloring G. For general graphs it is NP-hard to determine their chromatic number [12]. It is even NP-hard to decide if a planar graph is 3-colorable [11].

In 1977 László Lovász [15] found a new way to obtain lower bounds for $\chi(G)$ by using topological methods. He proved Kneser's conjecture, which he formulated as a question on the chromatic number of certain graphs, now called Kneser graphs. Lovász constructed a simplicial complex $N(G)$ that reflects the neighborhood relations in G. He showed that the connectivity of $N(G)$, which is a topological invariant, gives a lower bound to $\chi(G)$. Lovász's proof, which uses the Borsuk-Ulam Theorem [6], was the inspiration for many other "topological" proofs. (The first proof [4] taking up the idea of using the Borsuk-Ulam Theorem already appeared on the subsequent pages in the same journal.) For an introduction to topological methods in combinatorics we recommend Jiří Matoušek's book [16]. For a brief historical survey see the article by Mark de Longueville [9]. Several descendants of Lovász's neighborhood complex were defined and studied. Each yields some lower bound on the chromatic number. Jiří Matoušek and Günter M. Ziegler surveyed these bounds and their interrelations in [17]. For upper bounds to these lower bounds see [8] and [19].

Another prominent complex is the homomorphism complex $\mathrm{Hom}(H, G)$. This cell complex represents all homomorphisms from the graph G to H. For an introduction and further references see [14]. Eric Babson and Dmitry Kozlov [3] proved a conjecture by Lovász which infers a lower bound on $\chi(G)$ from the connectivity of $\mathrm{Hom}(C_{2r+1}, G)$, where C_{2r+1} is a cycle of odd length. Independent sets play an important role in these homomorphism complexes $\mathrm{Hom}(G, H)$. Hence the proof of Babson and Kozlov is another motivation to the study of independent sets.

In the context of simplicial complexes there are two main points of interest: the homotopy type or their homeomorphism types. Roughly speaking the homotopy type characterizes the number of holes of a topological space. The homeomorphism type specifies the space in much more detail. For example, the independence complex IC(G) in Figure 2 is homotopy equivalent to the sphere S^1, the circle. But it is not homeomorphic to S^1 because IC(G) is not purely 1-dimensional. Homeomorphism type "sees" the dimension. But it does not recognize bends; for example S^1 is homeomorphic to the boundary of a quadrilateral.

The homotopy types of the independence complexes of paths, cycles and trees are known. Louis J. Billera and Amy N. Myers [5] proved non-pure shellability of interval orders. This implies that the independence complex of each path is a wedge of spheres. Kozlov [13] showed that the independence complex of each cycle C_n is homotopy equivalent to a sphere or to a wedge of two spheres. This result was also proved by Manoj K. Chari and Michael Joswig [7] using discrete Morse functions. The following result is a corollary of Kozlov's theory of complexes of directed trees. It sharpens the result by Billera and Myers. Richard Ehrenborg and Gábor Hetyei generalized it in [10].

Proposition 1. *The independence complex* IC(T) *of a tree T either is contractible or it is homotopy equivalent to a sphere.*

This fact is the starting point of the construction suggested in the present work. We explain this construction with the help of the example illustrated in Figure 2. We start with the poset IP(T) of independent sets of T ordered by inclusion. We reverse this order and get the dual poset $\mathrm{IP}^{\mathrm{op}}(T)$, which is shown in Figure 3. Now we ask: "Is there a complex DIC(T) such that its face poset is $\mathrm{IP}(T)^{\mathrm{op}}$?"

At first sight the reader might think that the answer is "No". This is correct – there is no such complex. But let us have a closer look. Let us try to construct a dual independence complex although it is not possible. We proceed analogously to the construction of dual polytopes. We choose the maximal independent sets of T as the vertices of DIC(T). They are $\{1,3,4,6\}$, $\{1,3,5\}$, $\{2,5\}$, and $\{2,6\}$. So far so good. The next thing we need are the edges of DIC(T). A good candidate is the set $\{1,3\}$; it is incident to exactly two vertices namely $\{1,3,4,6\}$ and $\{1,3,5\}$. Thus we choose it as the edge between them. Because of the same reason we choose $\{6\}$, $\{2\}$, and $\{5\}$ as edges of DIC(T).

In contrast the set $\{3,4,6\}$ behaves completely differently. This set belongs to only one simplex, namely $\{1,3,4,6\}$. Thus $\{3,4,6\}$ cannot be an edge. On the other hand, we cannot make it a vertex because it is a subset of $\{1,3,4,6\}$. At this point we act like topologists. We say that the set $\{3,4,6\}$ is not essential for the topology of IC(T) – we simply ignore it. This set does not appear in the dual complex. Generally we omit all independent sets $\mathtt{I}$ whose $\mathrm{link}(\mathtt{I}, \mathrm{IC}(T))$ is contractible. In our example we cross out more than half of all the independent sets. What remains is the *dual independence poset* DIP(T), shown in Figure 3.

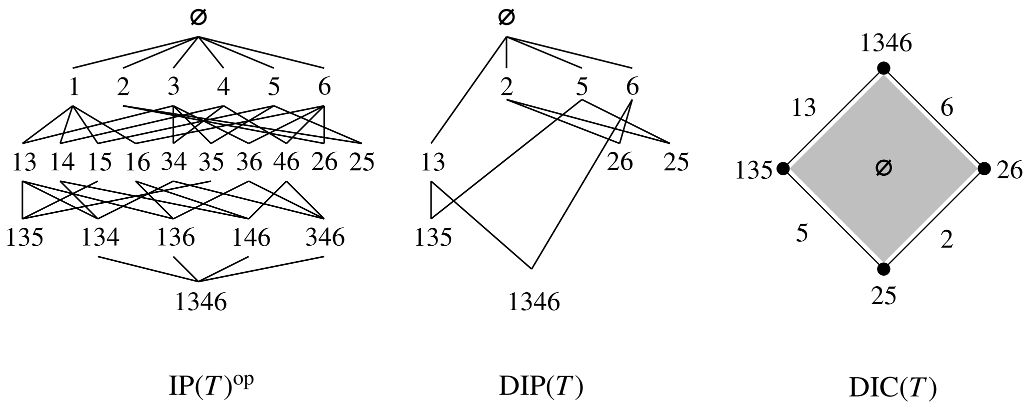

Figure 3: The face poset of the graph T and its dual

Now we have a second try and ask whether this poset DIP(T) is the face poset of a regular cell complex. This time the answer is "Yes." The poset DIP(T) is the face poset of the quadrilateral shown in Figure 3.

There is no reason why this construction should work for general graphs. Let us consider the mother of all counterexamples the Petersen Graph G_P. It is shown in Figure 4. Let p be an arbitrary node. Let $R := G_\mathrm{P} \setminus N[p]$ be the subgraph obtained from G_P by deleting p and its neighbors. It follows from a general principle that $\mathrm{link}(p, \mathrm{IC}(G_\mathrm{P})) = \mathrm{IC}(R)$. This link is homotopy equivalent to a wedge of two spheres S^1 as shown in Figure 4. In particular $\mathrm{link}(p, \mathrm{IC}(G_\mathrm{P}))$ is not contractible. Thus by construction the independent set $\{p\}$ is an element of the poset DIP(G_P).

In our imaginative dual independence complex this $\mathrm{link}(p, \mathrm{IC}(G_\mathrm{P})$ turns into the boundary of the cell $p \in \mathrm{K}$. More precisely p would be a cell whose boundary is homotopy equivalent to a wedge of two spheres. This is not possible. Hence the dual independence poset DIP(G_P) of the Petersen Graph is not the face poset of a cell complex.

This example shows that the dualization construction is restricted to a small class of independence complexes. Now Proposition 1 comes into play. It implies that links in the independence complexes of trees are either homotopy equivalent to spheres or they are contractible. Thus the "boundaries" of the potential cells of the dual complexes are homotopy equivalent to spheres. But we still need much more. We need that these boundaries are *homeomorphic* to spheres. The exciting result of this thesis is that for paths and cycles this condition is actually fulfilled. Moreover, it is true for forests whose components all are paths. We call such forests *p-forests*. (The letter p stands for "paths"). For the precise statement of

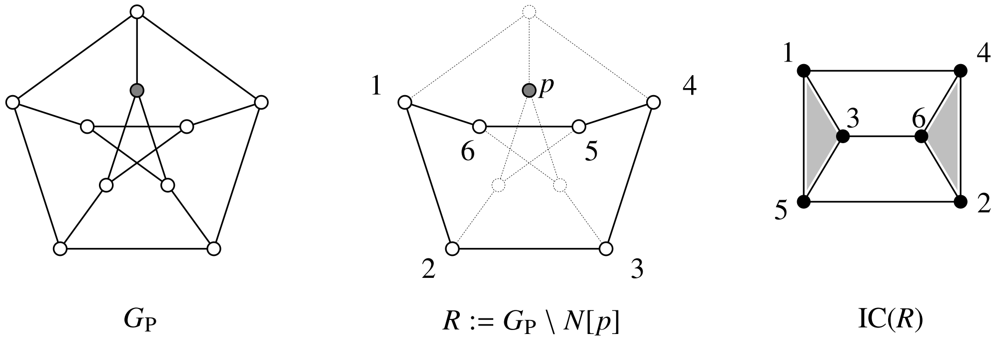

Figure 4: The Petersen Graph G_P, the complement of $N[p]$ and its independence complex.

our result we say a forest T is *spherical* if $\mathrm{IC}(T)$ is homotopy equivalent to a sphere. We define the size of a path of length ℓ to be $\lfloor \frac{\ell+2}{3} \rfloor$; the size of p-forest is the sum of the sizes of its components.

Theorem 2 (Dualization for paths). *For every path and more generally for every p-forest T there is a regular cell complex* $\mathrm{DIC}(T)$ *such that its face poset is the dual independence poset*

$$\mathcal{F}(\mathrm{DIC}(T)) = \mathrm{DIP}(T).$$

This cell complex $\mathrm{DIC}(T)$ *is a shellable ball of dimension* $\mathrm{size}(T)$. *If the p-forest T is spherical then the empty set $\varnothing$ is the maximal cell of* $\mathrm{DIC}(T)$*; its boundary is a shellable sphere.*

This construction generalizes to cycles C_n because the complement of an independent set in a cycle is a p-forest. With a slight modification of the definition (omitting $\varnothing$ as a maximal cell) the dual independence complex $\mathrm{DIC}(C_n)$ is homotopy equivalent to the non-dual complex $\mathrm{IC}(C_n)$. Thus $\mathrm{DIC}(C_n)$ is homotopy equivalent to a sphere or to a wedge of two spheres. Given these homotopy types the dual independence complexes turn out to be beautiful models of these types.

Theorem 3 (Dualization for Cycles). *For every cycle $C_n, n \geq 2$ there is a regular cell complex* $\mathrm{DIC}(C_n)$ *such that its face poset is the dual independence poset* $\mathrm{DIP}(T)$. *This complex is homeomorphic to the following spaces*

$$\mathrm{DIC}(C_n) \;\cong\; \begin{cases} S^{k-1} & \textit{if } n = 3k-1, \\ S^{k-1}_{3/2} & \textit{if } n = 3k, \\ S^{k-1} & \textit{if } n = 3k+1. \end{cases}$$

The space $S^k_{3/2}$ is homeomorphic to a union of three k-dimensional balls glued together along their boundaries via homeomorphisms to S^{k-1}. For example the dual independence complex of the cycle C_9 is drawn in Figure 5. It is homeomorphic to the the space $S^2_{3/2}$, a 2-sphere with a “third hemisphere”. It consists of 9 pentagons; each of them corresponds to one node of the graph C_9. Nodes at distance 3 and 6 correspond to pentagons that lie in the same “hemisphere”. Adjacent nodes in correspond to disjoint pentagons.

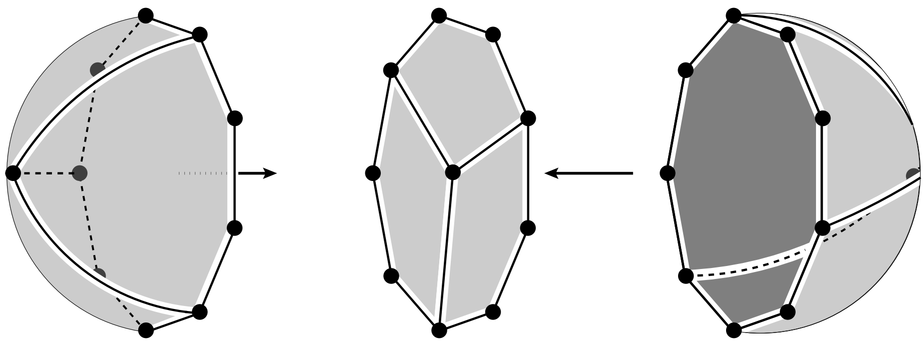

Figure 5: The dual independence complex of the cycle C_9. It is homeomorphic to the union of three 2-balls glued together along their boundaries.

In the following we sketch the proof of Theorem 2 and its generalizations to trees and forests. Let T be a forest (not necessarily a p-forest). In order to show that DIP(T) is the face poset of a regular cell complex we study its ideals $\mathrm{DIP}(T)_{\leq \mathtt{F}}$ For each ideal of this type there is a spherical forest S such that DIP(S) is isomorphic to it. Thus we focus on spherical forests. In particular we study the *facets* of spherical forests S, which are the coatoms of the posets DIP(S). As an example Figure 6 shows the facets of the path P_8. In order to understand the structure of these facets we classify the nodes of trees and forests into six different *types*: $\alpha^*, \alpha^\dagger, \beta^*, \beta^\dagger, \tau, \mu$. Given a node of a certain type we prove which types of neighbors it may have, which it must have, and which it cannot have. Based on this classification we group the nodes according to their types into *families*, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \ldots$ Then we build certain subtrees from them, so called *flocks*; for example $\mathfrak{ab}$. These notions enable us to describe the intricate structures of the facets in surprisingly simple terms.

Lemma 4 (Characterization of Facets). *Let S be a spherical forest. An independent set $\mathtt{F}$ is a facet if and only if either $\mathtt{F} = \{p\}$ for some α-node p or if $\mathtt{F}$ is the set of leaves of some flock.*

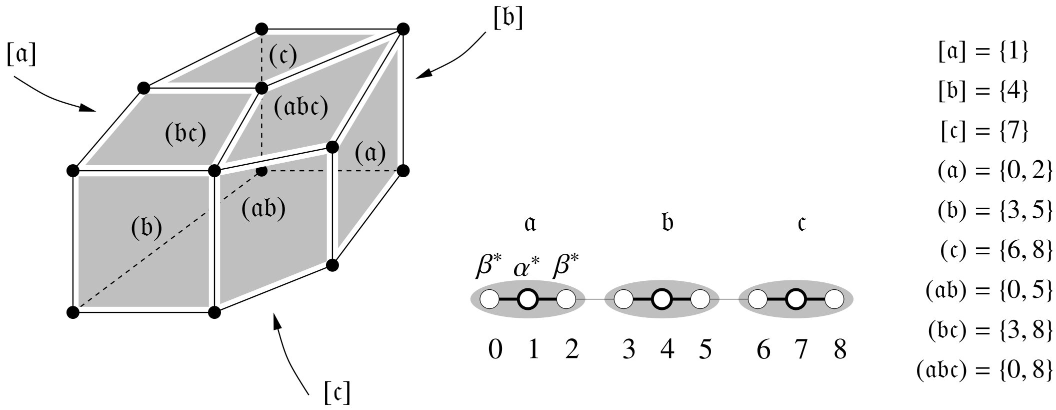

Figure 6: The dual independence complex of the path P_8 and its facets.

We use this characterization which is valid for trees and forests in order to show that the dual independence poset DIP(T) for every path T (and more generally every p-forest)

- has a rank function,
- has the diamond property, and
- admits a recursive coatom ordering,

These claims imply Theorem 2. We generalize some of these results to trees.

Theorem 5 (Rank and Diamonds on Trees). *Let T be a tree or more generally a forest. Its dual independence poset* DIP(T)

- *has a rank function, and*
- *has the diamond property.*

The dual independence posets of a path (and a tree) is not a semi-lattice, nor is it semimodular in general. For trees T the structure of DIP(T) can be even more intricate: its not even a pure poset in general. As a consequence, if T is a tree then the intersection of two principal ideals $\mathrm{DIP}(T)_{\leq \mathsf{F}} \cap \mathrm{DIP}(T)_{\leq \mathsf{G}}$ is not necessarily pure. This is the reason why we cannot easily transfer the recursive coatom ordering of paths to the realm of trees. Despite these difficulties we conjecture that also for trees there is a dual independence complex.

Conjecture 1 (Dualization for trees). *For every tree T, or more generally every forest, there is a regular cell complex* DIC(T) *such that its face poset is* DIP(T).

References

[1] David Applegate and Robert Bixby and Vašek Chvátal and William Cook, *On the solution of traveling salesman problems*, in International Congress of Mathematicians (Berlin 1998), vol. Extra Volume ICM 1998, Vol. III of Documenta Math., 1998, pp. 645–656.

[2] David Applegate and Robert Bixby and Vašek Chvátal and William Cook and Keld Helsgaun, *Optimal tour of Sweden*. `http://www.tsp.gatech.edu/sweden/index.html`.

[3] Eric Babson and Dmitry N. Kozlov, *Proof of the Lovász Conjecture*. preprint, 40 pages, 2004, `arXiv:math.CO/0402395`.

[4] Imre Bárány, *A short proof of Kneser's conjecture*, J. Combin. Theory Ser. A, **25** (1978), pp. 325–326.

[5] Louis J. Billera and Amy N. Myers, *Shellability of interval orders*, Order, **15** (1998), pp. 113–117.

[6] Karol Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fundamenta Mathematicae, **20** (1933), pp. 177–190.

[7] Manoj K. Chari and Michael Joswig, *Discrete Morse complexes*, in Proceedings of ACCOTA 2002, International Workshop Combinatorial and Computational Aspects of Optimization, Topology and Algebra, 2002. `arXiv:math.CO/0011234`.

[8] Péter Csorba, and Carsten Lange, and Ingo Schurr, and Arnold Wassmer, *Box complexes, neighborhood complexes, and the chromatic number*, J. Combin. Theory Ser. A, **108** (2004), pp. 159–168.

[9] Mark de Longueville, *25 years proof of the Kneser conjecture, the advent of topological combinatorics*, EMS-newsletter, (2004), pp. 16–19.

[10] Richard Ehrenborg and Gábor Hetyei, *The topology of the independence complex*. preprint, 2003.

[11] Michael R. Garey, and David S. Johnson, *Computers and intractability*, W. H. Freeman and Co., San Francisco, Calif., 1979.

[12] Karp, Richard M., *Reducibility among combinatorial problems*, in Complexity of computer computations (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972), Plenum, New York, 1972, pp. 85–103.

[13] Dmitry N. Kozlov, *Complexes of directed trees*, J. Comb. Theory Ser. A, **88** (1999), pp. 112–122.

[14] ——, *Chromatic numbers, morphism complexes, and Stiefel-Whitney characteristic classes*, in Geometric Combinatorics, vol. 14 of IAS/Park City Mathematics Series, Amer. Math. Soc., Providence, RI, 2005. in press.

[15] László Lovász, *Kneser's conjecture, chromatic number, and homotopy*, Journal of Combinatorial Theory, Series A, **25** (1978), pp. 319–324.

[16] Jiří Matoušek, *Using the Borsuk-Ulam theorem, Lectures on topological methods in combinatorics and geometry*, Universitext, Springer-Verlag, Berlin, 2003.

[17] Matoušek, Jiří and Ziegler, Günter M., *Topological lower bounds for the chromatic number: a hierarchy*, Jahresber. Deutsch. Math.-Verein., **106** (2004), pp. 71–90.

[18] Neil J. A. Sloane, *Challenge problems: Independent sets in graphs*. `http://www.research.att.com/~njas/doc/graphs.html`.

[19] James W. Walker, *From graphs to ortholattices and equivariant maps*, J. Combin. Theory Ser. B, **35** (1983), pp. 171–192.

[20] Günter M. Ziegler, *Lectures on Polytopes*, no. 152 in Graduate Texts in Mathematics, Springer-Verlag, Berlin, 1995.