Semester Report SS05 of Sarah Kappes

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Field of Research:	Discrete Mathematics
Topic:	Orthogonal Surfaces
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Research and Results

Orthogonal surfaces

Let $V \subset \mathbb{N}^d \subset \mathbb{R}^d$ be an antichain with respect to the dominance order. The *filter* generated by V is $\langle V \rangle = \{\alpha \in \mathbb{R}^d \mid \alpha \geq v \text{ for some } v \in V\}$. The boundary of $\langle V \rangle$ is an *orthogonal surface* S_V.

The surface S_V is called *generic* if two points of V that are no suspensions do not share any coordinates. In this case, the surface defines a simplicial complex, namely a *Scarf-complex*. This complex is always polytopal.

Orthogonal surfaces have been studied primarily for the generic case or for dimension three.

In the last few months I studied the *non-generic* case, which means that any two points of V can share coordinates. In particular, I was interested in the question which orthogonal surfaces give rise to polytopes. It turned out to be quite difficult to give a general definition of the *orthogonal complex*.

- The face inclusion order of the complex we are interested in is isomorphic to the dominance order on the *characteristic points* of S_V, so it is important to have a combinatorial definition for these points. If S_V is generic, every component-wise maximum of elements of V that is contained in S_V is a characteristic point. This is not true anymore for the non-generic case.

- If S_V is non-generic, there can be *degeneracies*. These are local structures that prevent a proper complex. For dimension three, there is basically one situation that causes problems. In higher dimensions, we need either a classification of all degeneracies, or preferably a combinatorial definition.

- In order to obtain a polytopal complex, the face-poset should be *graded*. For dimension three, this corresponds to the *rigidity* of the surface.

The tasks were to find general combinatorial definitions for the characteristic points of a surface as well as for the properties of degeneracy and rigidity.

Our starting point was the geometric definition for characteristic points. Based on this, we defined the non-degeneracy of a surface.

For non-degenerate surfaces, we then gave a combinatorial definition for characteristic points. In this case, we were also able to define the combinatorial *rank* of a characteristic point, in relation to its geometrical rank. For example, local minima have rank 0 and local maxima have rank $d-1$.

Finally, we required that characteristic points of the same rank should be incomparable. This corresponds to a generalization of the rigidity-criterion.

We then showed that - in contrast to dimension three - rigidity is not restrictive enough to guarantee that the resulting complex is polytopal. We constructed examples of rigid surfaces such that the dominance order on the characteristic points violates the *diamond-property* or the *lattice-property* or both.

Future work

- The examples show that non-degeneracy and rigidity is not enough to ensure that the orthogonal complex is polytopal. The next task is to find more restrictive criteria such that a surface satisfying them gives rise to a polytopal orthogonal complex. Currently, I work on some special constructions that avoid the problems of the counterexamples.

- It is an important question which simplicial polytopes can be realized as generic orthogonal complexes. Last year, I was able to find a combinatorial realization criterion for this case. It would be very interesting whether this criterion can be generalized to non-generic complexes, i.e. non-simplicial polytopes.

Activities

- Attended the "European Workshop on Computational Geometry" (EWCG) 2005 in Eindhoven

  Talk "On Pseudo-Convex Decompositions, Partitions, and Coverings"

- Attended the lecture "Diskrete Strukturen" by Prof. Felsner

- Attended the Noon Seminar of the workgroup "Diskrete Mathematik" at TU berlin

  Talk "What is an orthogonal complex?"

- Attended the seminar "Graphen Zeichnen" of the workgroup "Diskrete Mathematik"

- Attended the Monday lectures and colloquia of the CGC

Preview

- Summer School on Geometric Combinatorics in Vienna, July 18-29 2005

- European Conference on Combinatorics, Graph Theory and Applications (EuroComb), Berlin, Sep. 5-9 2005

- CGC-Workshop, Hiddensee, Sep. 25-28 2005

References

[1] Bayer, Peeva, Sturmfels: "Monomial Resolutions", Math. Res. Lett. 5 (1998), no. 1-2, 31-46.