Semester Report SS 05 of Stephan Hell

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Topic:	Topological Methods in Combinatorics and Geometry
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Results and Current Work

This semester I have spent most of my time on finishing a project started in the last semester under the supervision of Jiří Matoušek; for details I refer to my preprint [3]. As main result, I have obtained a topological fractional Helly theorem extending a result of Alon, Kalai, Matoušek, and Meshulam [1]. A brief summary of my results in [3] can be found in the next paragraph.

Fractional Helly. Helly's theorem is a classical theorem in convex geometry: For every finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ in which every $d+1$ or fewer sets have a common point we have $\bigcap \mathcal{F} \neq \emptyset$. The fractional Helly theorem for convex sets is a deep extension of it obtained by Katchalski and Liu in 1979: The family of convex sets in $\mathbb{R}^d$ has fractional Helly number $d+1$. Here a finite or infinite family $\mathcal{F}$ of sets has *fractional Helly number* k if for each $\alpha \in (0,1]$ there is a $\beta(\alpha) > 0$ such that the following implication holds: For all $F_1, F_2, \ldots, F_n \in \mathcal{F}$ such that $\bigcap_{i \in I} F_i \neq \emptyset$ for at least $\lfloor \alpha \binom{n}{k} \rfloor$ index sets $I \in \binom{[n]}{k}$, there exists a point which is in at least $\lfloor \beta n \rfloor$ of the sets F_i. Here $[n]$ is short for the set $\{1, 2, \ldots, n\}$, and $\binom{X}{k}$ for the set of k-element subsets of a set X.
There is two main tasks concerning fractional Helly theorems:

- For a family $\mathcal{F}$ find $\beta(\alpha) > 0$ optimal, as large as possible.
- Determine new families of sets that admit a fractional Helly theorem. What is their fractional Helly number?

I focused on the second problem motivated by a question of Kalai and Matoušek: *Is there a homological analog of VC-dimension?* In [3], I gave in some sense a positive answer to this question: Homological conditions imply a fractional Helly theorem.
Using a fractional Helly theorem for d-Leray families from [1], I have obtained the following result.

Theorem 1 (Topological fractional Helly theorem) *Let $\mathcal{F}$ be a finite family of open sets (or of subcomplexes of a cell complex) in $\mathbb{R}^d$, and $k \geq d$ such that for all subfamilies $\mathcal{G} \subseteq \mathcal{F}$ one of the following conditions holds:*

1. *$\bigcap \mathcal{G}$ is empty, or*

2. *the reduced homology groups of $\bigcap \mathcal{G}$ vanish in dimension at least $k - |G|$, that is*
$$\tilde{H}_n(\bigcap \mathcal{G}) = 0 \text{ for all } n \geq k - |\mathcal{G}|.$$

Then $\mathcal{F}^\cap$ has fractional Helly number $k+1$. Moreover, we can choose $\beta(\alpha) = 1 - (1-\alpha)^{1/(k+1)}$.

This result implies fractional Helly number $k+1$ for families satisfying a homological condition which depends on k. We call a family of sets satisfying conditions (i) and (ii) a $(k - |\mathcal{G}|)$-*acyclic* family.
For $k = d$ we obtain fractional Helly number $k+1$ in a more general setting than good covers. The case $k > d$ admits even more general families of sets in $\mathbb{R}^d$. The price one has to pay for increasing the intersection complexity of the F_i is a higher fractional Helly number. The proof uses a spectral sequence argument.
The (p,q)-theorem for convex sets was conjectured by Hadwiger and Debrunner, and proved by Alon and Kleitman in 1992. For this let p, q, d be integers with $p \geq q \geq d+1 \geq 2$. Then there exists a number $\mathrm{HD}(p,q,d)$ such that the following holds: Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ satisfying the (p,q)-condition; that is, among any p sets of $\mathcal{F}$, there are q sets with a non-empty intersection. Then $\tau(\mathcal{F}) \leq \mathrm{HD}(p,q,d)$, where $\tau(\mathcal{F})$ denotes the *transversal number* of $\mathcal{F}$, i. e. the smallest cardinality of a set $X \subseteq \bigcup \mathcal{F}$ such that $F \cap X \neq \emptyset$ for all $F \in \mathcal{F}$. It was observed in [1] that the crucial ingredient in the proof is a fractional Helly theorem for $\mathcal{F}^\cap$. Therefore the above topological fractional Helly theorem implies immediately a new (p,q)-theorem using the general tools developed in [1].

Theorem 2 ((p,q)-theorem for $(k - |\mathcal{G}|)$-acyclic families) *The assertions of the (p,q)-theorem also hold for finite $(k - |\mathcal{G}|)$-acyclic families of open sets (or of subcomplexes of a cell complex) in $\mathbb{R}^d$ where $p \geq q \geq k \geq d+1 \geq 2$.*

This result implies the (p,q)-theorem for good covers in [1].

Topological Tverberg. My preprint "On the number of Tverberg partitions in the prime power case" [4] was accepted for publication in the European Journal of Combinatorics. Currently I study a relaxed version of the

topological Tverberg problem: What happens if we increase the dimension of the simplex? Does the cohomological index first introduced by Fadell and Husseini [2] give new insight in this weaker setting for arbitrary q?
Moreover, I will finish my computer project on counting the number of Tverberg partitions for randomly generated maps in the plane and on the two-dimensional sphere. Up to now, all results confirm Sierksma's conjecture in dimension 2. For this, I have studied several classes of drawings of K_{3q-2} in previous semesters.

Activities

- Attended Lectures and Colloquia of the CGC
- Talk at CGC Colloquium, May 30, TU Berlin
- Attended CGC Spring School *Enumerative Combinatorics*, June 1–4, Netzeband
- Talk at *Noon seminar* of our group Discrete Geometry, June 21, TU Berlin
- Attended lecture *Diskrete Geometrie II* of Günter M. Ziegler, TU Berlin
- Talk at seminar *Graphen zeichnen* of Stefan Felsner, July 8–10, Bad Freienwalde

Preview

- Attend Summer School *Geometric Combinatorics*, July 18–29, Vienna
- Talk *On a topological fractional Helly theorem* at Summer School *Geometric Combinatorics*, Vienna
- Attend EuroComb 05, September 5–9, TU Berlin

References

[1] N. Alon, G. Kalai, J. Matoušek, and R. Meshulam, *Transversal numbers for hypergraphs arising in geometry*, Adv. Appl. Math. **29** (2003), pp. 79–101.

[2] E. Fadell and S. Husseini, *An ideal–valued cohomological index theory with applications to Borsuk–Ulam and Bourgin–Yang theorems*, Ergod. Th. & Dynam. Sys. **8*** (1988), pp. 73–85.

[3] S. Hell, *On a topological fractional Helly theorem*, 2005. Preprint arXiv.math.CO/0506399.

[4] S. Hell, *On the number of Tverberg partitions in the prime power case*, Europ. J. of Combinatorics (2005). to appear.