Semester Report SS 04 of Arnold Waßmer

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Field of Research and Results

After my time in the CGC-program I am about to finish my thesis. It includes the following topics
– A discrete version of the Borsuk-Ulam Theorem,
– the symmetries of cyclic polytopes,
– a dual independence complex of paths, trees and cycles

In the following I will explain some details of the last topic. The independence complex $\texttt{Ind}(G)$ of a graph G is the simplicial complex of the independent sets of G. Thus the vertex set of $\texttt{Ind}(G)$ is the node set $V(G)$. A set of vertices forms a simplex if the corresponding node set does not contain two neighbor nodes. In general this complex $\texttt{Ind}(G)$ is not pure, i.e. its inclusion maximal simplices vary in size.

For tree graphs T Dmitry Kozlov [3] proved that $\texttt{Ind}(T)$ either is contractible or is homotopy equivalent to a sphere. As a consequence of that the links of simplices $\sigma \in \texttt{Ind}(T)$ either are contractible or are homotopy equivalent to a sphere. In the latter case we say the independent set σ is *spherical*. For a tree T let $S(T)$ be the poset of all spherical independent sets and let $\texttt{DIP}(T)$ be the poset dual to $S(T)$.

I could prove that for every tree T there is a regular cell complex such that its face poset is $\texttt{DIP}(T)$. This complex is the *dual independence complex* $\texttt{DIC}(T)$. Moreover if the (primal) independence complex $\texttt{Ind}(T)$ is homotopy equivalent to a sphere then the boundary of the dual $\texttt{DIC}(T)$ is *homeomorphic* to a sphere of the same dimension.

For example there is a tree T such that its independence complex $\texttt{Ind}(T)$ is the boundary of the octahedron with one full tetrahedron attached to one of the triangles. Dualizing yields the boundary complex of a cube as $\texttt{DIC}(T)$. In this sense dualizing yields smaller and nicer independence complexes.

The boundaries of the cells of $\texttt{DIC}(T)$ are not only homeomorphic to spheres moreover they are constructible. Hence they are PL-spheres.

On the way to the proof of this result one has to study the coatoms of $\mathtt{DIP}(T)$ for spherical trees T. These coatoms correspond to the facets of the complex $\mathtt{DIC}(T)$. They are inclusion minimal non-empty spherical independent sets.

These facets turn out to fall into two classes. Facets of the first kind, called α-facets, contain one node only. Facets of the second kind, β-facets contain at least two nodes. Analogously one can classify the nodes of the graph T into α or β in a natural way. It turns out that α-facets contain only one α-node and β-facets only contain β-nodes. More generally it turns out that computing the dimension of a cell is done by counting certain α-nodes. Moreover this classification of the facets into α and β-type facets helps to prove the diamond property for the poset $\mathtt{DIP}(T)$.

The complex $\mathtt{DIC}(T)$ turns out to be a subdivision of the boundary complex of a cube, which is shellable. A related but weaker notion of shellability is constructability, which translated to the world of posets is called *recursive dividability* (see[2]).

I could construct a recursive division of the poset $\mathtt{DIP}(T)$ for all spherical trees T. Together with the diamond property of $\mathtt{DIP}(T)$ it implies that $\mathtt{DIP}(T)$ is a CW-poset; hence it is the face poset of some regular CW-complex $\mathtt{DIC}(T)$ (see[1]). This proves via some poset argumentation that the dual independence complex $\mathtt{DIC}(T)$ exists for every tree T, moreover it exists for every forest. Since a non empty independent set cuts a cycle graph into a forest this theorem easily extends to cycles.

I want to take the opportunity to thank all members of the graduate program who made this work possible: the coordinators Bettina Felsner and Andrea Hoffkamp, the speaker of the program Helmut Alt and my supervisor Günter M. Ziegler. Thank you very much!

References

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