Semester Report SS04 of Ares Ribó Mor

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Field of Research:	Geometry and Combinatorics
Topic:	Self-Touching Configurations and Rigidity Theory Spanning trees of Planar Graphs Counting Polyominoes Map Foldability
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Field of Research and Results

Counting polyominoes on the plane

This semester I have been mostly working in counting polyominoes, together with Günter Rote and Gill Barequet.

A polyomino of size n is a connected set of n adjacent squares on a regular square lattice. The symbol $A(n)$ denotes the number of fixed polyominoes of size n on the plane.

The problem of counting polyominoes has received a lot of attention, but to this day there is no known analytic formula for $A(n)$. The only known methods for computing $A(n)$ are based on explicitly or implicitly enumerating all polyominoes.

We are based on the Jensen transfer-matrix algorithm ([5],[6]), an improved version of the algorithm of Conway and Guttman [3]. Using his algorithm, which was also optimised by Knuth [8], Jensen computed $A(n)$ for $n \leq 48$. Later, he also claimed in an unpublished work to obtain till $n \leq 56$.

It is known that $A(n)$ is exponential in n. Klarner [7] showed that $A(n) \sim C\lambda^n n^\theta$ (for some constants $C > 0$ and $\theta \approx -1$), so that the limit $\lambda = \lim_{n\to\infty} A(n+1)/A(n)$ exists. Golomb [4] gave λ its name *Klarner's constant*, of which not even a single significant digit is known for sure. There have been several attempts to lower and upper bound λ, as well as to estimate it, based on knowing $A(n)$ up to certain values of n. The best known lower and upper bounds were 3.903184 and 4.649551. The constant λ is estimated to be around 4.06 [3].

We improve the lower bound on Klarner's constant, counting polyominoes on a different grid structure: the twisted cylinder. The *twisted cylinder* of width W is obtained from the integer grid $\mathbf{N} \times \mathbf{N}$ by identifying point (i, j)

with $(i+1, j+W)$, for all i, j. We fix the width W, and the length can be arbitrarily large. This "twist" is that it allows to build up the cylinder incrementally by a *uniform* process, adding one new cell at a time. This leads to much simpler recursion and algorithm.

We prove a bijection between the set of signatures of length W and the set of Motzkin paths of length $W+1$. We have implemented a program in C which iterates the transfer equations. Our program encodes and ranks the states by Motzkin paths, which is much more space-efficient than the previous encodings found in the literature.

We have obtained a lower bound of 3.980137 for $W = 22$.

But for $W = 23$, the number of Motzkin paths of length 24 is $M = 2^{31.57}$, and this requires about 48 GigaBytes of memory. We could try to optimise this memory requirement doing some programming tricks, but still we do not think that we could reach the historical lower bound of 4. The running time grows approximately by a factor of 3 when increasing W by one unit.

We are currently giving the finishing touches to our joint paper.

Locked and unlocked polygons

We have restarted a discussion that took place in the summer semester 2002, together with Günter Rote, Bob Connelly, Erik Demaine, Martin Demaine, Sandor Fekete and Joseph Mitchell, about locked and unlocked chains of triangles, squares, and other simple polygons.

We have found several examples of locked and unlocked chains of equilateral triangles of the same or different size, also acute triangles, or rectangles of different sizes, and examples of unlocked chains of squares.

Some locked examples can be proved to be infinitesimally locked via LP as described in [2], and hence locked. We implemented some routines in AMPL (a modelling language for linear programming) which show the stresses of an infinitesimally locked self-touching configuration. Similarly, we can prove a configuration to be infinitesimally unlocked via LP. But we have examples of infinitesimally flexible configurations which are locked and they can not be opened by a real motion. We constructed some of these examples with Cinderella and checked if the directions of movement given by LP give a real opening.

We have proven that if we "thicken" each edge of a polygonal chain by adding right isosceles triangles to each side, the resulting chain of squares never self-intersects under expansive motions of the underlying linkage. The same is true if we thicken each edge by taking a union of disks centered at points on the edge. Actually we believe that every segment pq in the chain may be replaced by a convex shape where every normal of the convex shape intersects the interior of the segment pq.

These results will be summarised in a joint paper this month.

Number of Spanning Trees of a Planar Graph

Last semester, we started to work on bounding the number T of spanning trees of a planar graph with n vertices, together with Günter Rote.

For lower bounds, we introduced a new method based on transfer matrices for enumerating T for recursively constructible families of graphs. We implemented the method in Maple, obtaining the lower bound of 5.029^n. For upper bounds, we could show that $T \leq 5.333^n$.

I am currently working on improving the upper bounds that we found last semester, for graphs without triangles ($T \leq 4^n$) and graphs without triangles and quadrilaterals ($T \leq 2.924^n$).

Maxwell-Cremona theorem

We are also finishing the joint paper with Günter Rote on the generalisation of the *Maxwell-Cremona theorem* for self-touching configurations, mentioned in the previous semester report.

Activities

- Attended the *Monday Lectures and Colloquia* of the Graduate Program.
- Attended the *Mittagsseminar Theoretische Informatik* at FU Berlin. Presentation of the talks:
  - *Counting polyominoes in the plane. A new lower bound for the Klarner's constant.*, March 30th, 2004.
  - *Schlegel Diagrams for 4-Polytopes*, June 3rd, 2004.
- Attended the Blockcourse *Discrete Geometry - Polytopes and More*, by Prof. Günter Ziegler, from April 20th to May 19th 2004, TU Berlin.
- *Berliner Algorithmen Tag 2004*, July 12th, 2004, TU Berlin.

References

[1] G. Barequet and M. Moffie, The complexity of Jensen's algorithm for counting polyominoes, *Proc. 1st Workshop on Analytic Algorithmics and Combinatorics*, New Orleans, LA, January 2004.

[2] R. Connelly, E. Demaine, G. Rote, *Infinitesimally locked self-touching linkages with applications to locked trees*, "Physical Knots: Knotting, Linking, and Folding Geometric Objects in $\mathbb{R}^3$". Contemporary Mathematics 304, American Mathematical Society 2002, 287–311.

[3] A.R. Conway and A.J. Guttmann, On two-dimensional percolation, *J. Physics, A: Mathematical and General*, 891–904, 28 (1995).

[4] S.W. Golomb, Polyominoes, 2nd ed., Princeton Univ. Press, 1994.

[5] I. Jensen, Enumerations of lattice animals and trees, *J. of Statistical Physics*, 102 (2001), 865–881.

[6] I. Jensen, `http://sunburn.stanford.edu/~knuth/programs/jensen.txt` (a personal communication with D.E. Knuth).

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[8] D.E. Knuth, `http://sunburn.stanford.edu/~knuth/programs.html#polyominoes` (a personal WWW page).

[9] J. Richter-Gebert, *Realization spaces of polytopes*, chapter 13. Lecture Notes in Mathematics **1643**, Springer–Verlag Berlin Heidelberg 1996.