Semester Report SS04 of Sarah Renkl

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Field of Research:	Discrete Mathematics
Topic:	Orthogonal Surfaces
PhD Student	at the program since October 2003

Field of Research

Let $V \subset \mathbb{N}^d \subset \mathbb{R}^d$ be an antichain with respect to the dominance order. The *filter* generated by V is $\langle V \rangle = \{\alpha \in \mathbb{R}^d \mid \alpha \geq v \text{ for some } v \in V\}$. The boundary of $\langle V \rangle$ is an *orthogonal surface* S_V in dimension k.

Let $A \subset V$. $\bigvee(A)$ is the component-wise maximum of all elements in A. A is *k-generating* if $|A| = k+1$ and $\bigvee(A - \{a\}) < \bigvee(A)$ for all $a \in A$. $F_A = V \cap Down(\bigvee(A))$ is called a *k-face*.

The main object under investigation is the complex of faces generated by V. This structure is called the *orthogonal complex*. With every face F, we have a characteristic point $\bigvee_{a \in F} a = \bigvee F$ on the surface S_V. This mapping embeds the containment order of the complex into the dominance order of $\mathbb{N}^d$, hence, the order dimension is $\leq d$.

The orthogonal complex is an *orthogonal triangulation*, if every k-face consists of exactly $k+1$ vertices. This is the case if and only if the elements of the antichain are in general position.

For $d = 3$, this results directly in the theory of Schnyder-colorings of 3-connected planar graphs. The orthogonal complex on a surface of dimension 3 is a 3-connected planar graph with the additional structure of a Schnyder-coloring. (Actually, this is only true if we forbid some degenerate subconfigurations.)

The Theorem of Schnyder states that the planar graphs are exactly the graphs with order dimension 3. By the Theorem of Steinitz, the 3-connected planar graphs are exactly the edge-graphs of 3-polytopes.

An orthogonal complex in dimension 4 consists of vertices, edges, 2-faces and 3-faces. It is not at all obvious whether it has similarly nice properties as in dimension 3.

Results

I have found some results that indicate that the orthogonal complexes of dimension 4 have a polytopal structure.

- For example, I was able confirm the conjecture that all graphs (as a substructure of the orthogonal complex) are 4-connected. That is a necessary condition for the graphs of 4-polytopes by the Theorem of Balinski.
- Furthermore, I was able to prove that Barnette's topological 3-sphere is not realizable on any orthogonal surface of dimension 4. This suggests that the faces are also globally well behaved.
- Another result states that every orthogonal triangulation is a simplicial complex with nice properties. For example, every 2-face is contained in exactly 2 3-faces etc.

Up to the facts stated above, it is still unresolved what relation there is between orthogonal complexes and polytopes. I hope that I will be able to complete the picture in some of the following aspects.

- It seems likely that the simplical complexes obtained from orthogonal triangulations are indeed the face-complexes of simplicial polytopes. I have started to investigate examples of small simplicial 4-polytopes to get some clues about their realization as orthogonal complexes.
- The generalization to polytopal complexes when the points of the antichain share coordinates is more complicated. In general, we will have degenerate subconfigurations so that the mapping between faces and generating subsets of V is not one-to-one. The task is to determine additional requirements that assure that the lattice property is maintained.
- It would also be very interesting to find a small 4-polytope that cannot be realized as an orthogonal complex. So far, it is only known that neighborly polytopes with more than 12 vertices cannot be generated, a fact that is due to the dimension of their incidence order.

Activities

- CGC-Doccourse in Prague, January 2004 - March 2004
  - Attended the course “Permutation groups” of Prof. Peter Cameron
  - Attended the course “Arrangements in computational and combinatorial geometry” of Prof. Micha Sharir
- Talk “Orthogonal Surfaces” at the COMBSTRU-Workshop at Bordeaux, (April 1st-3rd 2004).
- Attended the CGC-Doccourse “Discrete Geometry” of Prof. Günther Ziegler in Berlin, April 20th - Mai 19th, 2004
- Attended the lecture “Graphen, Zufall, Algorithmen” of Prof. Felsner
- Attended the seminar “Geometrische Aspekte der Graphentheorie” of Prof. Felsner
- Attended the Monday Lectures and Colloquia of the CGC
- Talk “Orthogonal Surfaces” at the “Doktoranden-und Diplomanden-seminar” of the Workgroup “Combinatorial Optimization and Graph Algorithms” of Prof. Möhring

Preview

- CGC-Workshop in Stels, October 4th-7th 2004
- Symposium Diskrete Mathematik at the ETH Zurich, October 7th-8th 2004