Semester Report SS04 of Andreas Paffenholz

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Field of Research:	Discrete Geometry
Topic:	Flag Vectors of 4-Polytopes
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Field of Research and Results

During this summer semester 2004 I hardly spent any time in Berlin, as I participated in the two block courses in Prague and then moved to Zurich for my "long-stay".

I attended the two block courses of the "Doc course Berlin-Prague 2004" that were held in Prague from January to March. The first course was by Peter Cameron on "Permutation Groups" and the second by Micha Sharir on "Arrangements". In March I returned to Berlin for three weeks before I moved to Zurich for my "long-stay" in the beginning of April. In Zurich I am a guest in the research group "Theoretical Computer Science" of Prof. Emo Welzl at the ETH. I will return to Berlin in the end of August.

The papers with G.M. Ziegler [PZ03] and A. Björner, J. Sjöstrand, and G.M. Ziegler [BSPZ04], that we submitted last year, were both accepted for publication in the beginning of this year. Both papers will appear in Discrete and Computational Geometry. I spent some time while I was in Berlin in March with the corrections suggested by the referees. Preparing for Zurich took almost all remaining time in Berlin.

In Zurich I continued with the work on my paper [Paf04a] about the application of the "E_t-construction" (as introduced in [PZ03]) to products of polytopes and gave a talk on this in the Mittagsseminar. Last semester I found a nice and simple "product" construction of $E(P \times Q)$, the E_{d-2}-construction applied to a product of two polytopes P and Q (where d is the dimension of the product), provided that we can construct $E(P)$ and $E(Q)$. To obtain $E(P \times Q)$ from these two "factors" one then performs some kind of "orthogonal product". The main application of this construction is a family E_{mn} of self-dual, 2-simplicial and 2-simple 4-polytopes with some interesting properties. The E_{mn} arise from the construction applied to a product of an m- and an n-gon. The most well-known member of this family is E_{44}, which is the 24-cell. I provided a simple and flexible construction for geometric realisations of these polytopes. In the last months I concentrated on realisation spaces and on combinatorial and geometric symmetries of the family E_{mn} of polytopes.

I proved that the projective realisation space of E_{33} is at least nine-dimensional. The proof of this fact explicitely parametrises a subset of the realisation space. For the polytope E_{44} I constructed a 4-parameter family of realisations proving that its realisation space is at least 4-dimensional. However, I also constructed realisations of E_{33} and E_{44} that cannot be obtained by my "product" construction.

After this I looked at the combinatorial and geometric symmetry groups of E_{mn}. I can prove that the existence of a certain geometric symmetry in a realisation of E_{mn} implies that it can be obtained from the "product" construction and that the two 2-dimensional factors must have some kind of symmetry. But as this symmetry in a 2-dimensional factor can only be realised for small m and n, we get the following theorem:

Theorem. *For all polytopes E_{mn}, $m, n \geq 3$, the product $S := \mathbb{Z}_m \times Z_n$ of two cyclic groups is a subgroup of the combinatorial symmetry group. However, for $(m,n) \notin \{(3,3),(3,4),(3,5),(3,6),(4,3),(4,4),(5,3),(6,3)\}$ the group S can never be a subgroup of the geometric symmetry group for any geometric realisation of E_{mn}.*

For all relatively prime $m, n \geq 5$ the combinatorial symmetry group of E_{mn} has an element that cannot be realised as a geometric symmetry for any geometric realisation of E_{mn}.

A polytope for which the combinatorial and geometric symmetry group differ was first obtained by J. Bokowski, G. Ewald, and P. Kleinschmidt [BEK84]. They constructed a 4-polytope P on ten vertices and a combinatorial symmetry of of P such that no geometric realisation of P has a geometric symmetry inducing it.

Last month I also started to work on a different problem. For the "Open Problem Workshop" in Cumpadials every participant was asked to submit some open (mathematical) problems that could be worked on in small groups during the workshop. While preparing my suggestions I got interested in the following problem: Which d-polytopes P have the property that there is a $(d+1)$-polytope $\tilde{P}$ that has only facets combinatorially equivalent to P? Polytopes with this property are called "facets", all others are "nonfacets".

I found this question in a paper of M. Perles and G.C. Shephard [PS67b], where they present some partial results mostly based on their work on angle sums in polytopes [PS67a]. Some more results are in a paper by D. Barnette [Bar80]. In particular Perles and Shephard try to classify the regular polytopes into facets and nonfacets. They succeed in all but five cases. These open cases are the icosahedron, the 24-cell, the 120-cell and the four and five dimensional cross polytopes. Using methods from metric geometry I prove in [Paf04b] that all these remaining cases are nonfacets. My methods probably also apply to some other classes of polytopes.

Finally I also continued my work on restrictions for k-cubical polytopes [JLP04], but that will mainly have to wait until I am back in Berlin. In the next time, and in particular after I have returned to Berlin I will have to concentrate on writing up my thesis. In the last weeks I have started writing up some introductory material.

Activities

Talks, Classes, Workshops:

- block courses ("Doc course Berlin-Prague 2004") in Prague from Mid-January to the beginning of March.
- Attended the class on "Random Graphs" by Prof. Bela Bollobas, Nachdiplom Lectures, ETH Zürich
- Mittagsseminar "Theoretical Computer Science", Institute of Theoretical Computer Science, ETH Zürich
- Oberwolfach Seminar "Discrete Computational Geometry", Oberwolfach, May 30 – June 5, 2004
- "Second GREMO Workshop on Open Problems", Cumpadials, Switzerland, July 4 – 8, 2004
- Talk on "Polytopes from Products," April 2004, Mittagsseminar "Theoretical Computer Science", Institute of Theoretical Computer Science, ETH Zürich

Forthcoming Activities:

- "DMV-Tagung", Heidelberg, September 12 – 17, 2004
- "CGC Annual Workshop", October 4 –7, 2004, Stels, Switzerland

Preview

I will be back in Berlin for the winter semester. As this is my last semester in the CGC, I will mostly be concerned with writing up my thesis.

References and Publications

[BSPZ04] Anders Börner, Jonas Sjöstrand, Andreas Paffenholz, and Günter M. Ziegler, *Bier spheres and posets*, to appear in Discr. Comp. Geom., available at `arXiv:math.CO/0311356`, April 2004, 15 pages.

[JLP04] Michael Joswig, Carsten Lange and Andreas Paffenholz, *Obstructions in Cubical Polytopes*, in preparation, March 2004.

[Paf04a] Andreas Paffenholz, *New polytopes from products*, in preparation, Zurich, July 2004.

[Paf04b] Andreas Paffenholz, *Regular Polytopes: Facets and Nonfacets*, in preparation, Zurich, July 2004.

[PZ03] Andreas Paffenholz and Günter M. Ziegler, *The E_t-Construction for Lattices, Spheres and Polytopes*, to appear in Discr. Comp. Geom. (Billera Festschrift) , available at `arXiv:math.MG/0304492`, March 2004, 20 pages.

[Bar80] David Barnette Nonfacets for shellable spheres. *Israel J. Math. 35* (1980), 286–288.

[BEK84] Jürgen Bokowski, Günter Ewald, and Peter Kleinschmidt, *On combinatorial and affine automorphisms of polytopes*, Israel J. Math. **47** (1984), 123–130.

[PS67a] Micha A. Perles and Geoffrey C. Shephard Angle sums of convex polytopes. *Math. Scand. 21* (1967), 199–218 (1969).

[PS67b] Micha A. Perles and Geoffrey C. Shephard Facets and nonfacets of convex polytopes. *Acta Math. 119* (1967), 113–145.