Semester Report SS04 of Daniela Kühn

Name:	Dr. Daniela Kühn
Supervisor(s):	Prof. Martin Aigner
Field of Research:	Discrete Mathematics
Topic:	Graph Theory
Postdoc	at the program since 1 October 2003

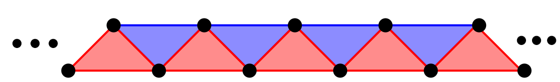
Fields of current Research & Results

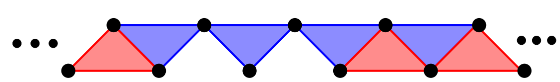
Hamilton cycles in 3-uniform hypergraphs

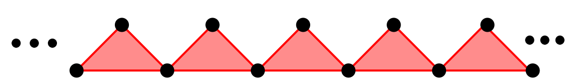
A classical theorem of Dirac states that every graph on n vertices with minimum degree at least $n/2$ contains a Hamilton cycle. If one seeks an analogue of this result for uniform hypergraphs, then several alternatives suggest themselves. In the following, we will restrict ourselves to 3-uniform hypergraphs $\mathcal{H}$. Thus each hyperedge of $\mathcal{H}$ consists of precisely 3 vertices.

A natural way to extend the notion of the minimum degree of a graph to that of a 3-uniform hypergraph $\mathcal{H}$ is the following. Given two distinct vertices x and y of $\mathcal{H}$, the *neighbourhood* $N(x,y)$ *of* x *and* y in $\mathcal{H}$ is the set of all those vertices z which form a hyperedge together with x and y. The *minimum degree* $\delta(\mathcal{H})$ is defined to be the minimum $|N(x,y)|$ over all pairs of vertices of $\mathcal{H}$.

A 3-uniform hypergraph $\mathcal{C}$ is called a *tight cycle* if $\mathcal{C}$ has at least 4 vertices and if there is a cyclic ordering of the vertices of $\mathcal{C}$ such that every 3 consecutive vertices form a hyperedge of $\mathcal{C}$ and $\mathcal{C}$ contains no other hyperedges. Note that the cyclic ordering of the vertices of $\mathcal{C}$ induces a cyclic ordering of the hyperedges of $\mathcal{C}$.


tight cycle


unconstrained cycle


loose cycle

A 3-uniform hypergraph $\mathcal{C}$ is an *unconstrained cycle* if it is a spanning subhypergraph of a tight cycle such that every vertex lies in at least one hyperedge of $\mathcal{C}$ and consecutive hyperedges of $\mathcal{C}$ share at least one vertex. An unconstrained cycle $\mathcal{C}$ is a called *loose* if there exists no unconstrained cycle which has the same number of vertices as $\mathcal{C}$ but fewer edges. Thus if the number n of vertices in a loose cycle $\mathcal{C}$ is even, then consecutive hyperedges in $\mathcal{C}$ have exactly one vertex in common and the number of hyperedges in $\mathcal{C}$ is exactly $n/2$. If n is odd, then exactly one pair of consecutive hyperedges in $\mathcal{C}$ have two vertices in common, all other consecutive pairs share only one vertex and the number of hyperedges in $\mathcal{C}$ is exactly $\lceil n/2 \rceil$.

The following result is joint work with Deryk Osthus [4]:

Theorem 1 *For every $\varepsilon > 0$ there is an integer $n_0 = n_0(\varepsilon)$ such that every 3-uniform hypergraph $\mathcal{H}$ with $n \geq n_0$ vertices and minimum degree at least $n/4 + \varepsilon n$ contains a loose Hamilton cycle (i.e. a loose cycle which contains all the vertices of $\mathcal{H}$).*

The bound on the minimum degree in Theorem 1 is best possible up to the error term εn. In fact, there are 3-uniform hypergraphs with minimum degree $n/4 - 1$ which do not even contain an unconstrained Hamilton cycle.

Recently, Rödl, Ruciński and Szemerédi [7] proved that if the minimum degree is at least $n/2 + \varepsilon n$ and n is sufficiently large, then one can even guarantee a tight Hamilton cycle. Their bound is also best possible up to the error term εn. The proofs of both our Theorem 1 and the result in [7] rely on the regularity lemma for 3-uniform hypergraphs due to Frankl and Rödl [2]. However, in [7] the authors make extensive use of the fact that the intersection of the neighbourhoods of any two pairs of vertices is nonempty, which is far from true in our case. For this reason, our argument has a rather different structure. (In fact, if n is divisible by 4 and we assume that our hypergraph has minimum degree at least $n/2 + \varepsilon n$, then the analogue of Theorem 1 is considerably easier to prove.)

Very recently, the regularity lemma for 3-uniform hypergraphs was generalized to k-uniform hypergraphs independently by Rödl and Skokan and by Gowers. This opens up the possibility of generalizing Theorem 1 to k-uniform hypergraphs.

Topological cliques in graphs of girth at least 66

Bollobás and Thomason [1] as well as Komlós and Szemerédi [8] independently proved that there exists a constant c such that every graph G of minimum degree $\geq cr^2$ contains a subdivision of the complete graph K_r on r vertices. This is best possible up to the value of the constant c. However, Mader [6] showed that if the girth of G is large, then a minimum degree of $r-1$ will do. Since the branch vertices in a subdivision of K_r have degree $r-1$, the bound on the minimum degree is tight. Mader's bound on the required girth was linear in r, but Deryk Osthus and I proved in [3] that a girth of 186 already suffices. Based on recent results of Thomas and Wollan [9], we [5] were now able to improve the bound on the girth further to 66. This implies that the conjecture of Hajós that every graph of chromatic number $\geq r$ contains a subdivision of K_r (which is false in general) is true for graphs of girth ≥ 66.

Activities

- talk "Substructures in graphs" at Birmingham University (February 2004)
- talk "Matchings in hypergraphs" in the Seminar "Combinatorics and Geometry" at FU Berlin
- I lectured a course on "Combinatorial Structures" at FU Berlin

Preview

I accepted an offer of a permanent lectureship at Birmingham University and will start there in September 2004.

References

[1] B. Bollobás and A. Thomason, Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs, *Eur. J. Comb.* **19** (1998), 883–887.

[2] P. Frankl and V. Rödl, Extremal problems on set systems, *Random Struct. Algorithms* **20** (2002), 131–164.

[3] D. Kühn and D. Osthus, Topological minors in graphs of large girth, *J. Combin. Theory B* **86** (2002), 364–380.

[4] D. Kühn and D. Osthus, Loose Hamilton cycles in 3-uniform hypergraphs of high minimum degree, in preparation.

[5] D. Kühn and D. Osthus, in preparation.

[6] W. Mader, Topological subgraphs in graphs of large girth, *Combinatorica* **18** (1998), 405–412.

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[8] J. Komlós and E. Szemerédi, Topological cliques in graphs II, *Comb. Probab. Comput.* **5** (1996), 79–90.

[9] R. Thomas and P. Wollan, An improved linear edge bound for graph linkages, preprint 2004.