Semester Report SS04 of Oliver Klein

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Field of Research:	Computational Geometry and Combinatorics
Topic:	Matching Shapes with a Reference Point
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Field of Research

Given two sets $A, B \in \mathcal{C}^2$, where $\mathcal{C}^2$ is the set of compact convex subsets of $\mathbb{R}^2$ (called figures), one can be interested in how these sets resemble each other. One good measure of resemblance is the Hausdorff-Distance δ_H. This distance is defined as the smallest ε such that the Euclidian distance from every point of A to its nearest point of B is at most ε and vice versa. For measuring the resemblance of the two sets, one has to determine the minimal Hausdorff-Distance under a given set of transformations. For example, these transformations can be translations, rigid motions (translations and rotations) or even similarity transformations (rigid motions and scalings). Algorithms for determining the optimal transformation to minimize the Hausdorff-Distance are known in all three cases, but the running-time of these algorithms is not satisfying for most applications.

To decrease the run-time, the authors of [1] use reference points to get an approximation for the problem. A reference point is a mapping $r : \mathcal{C}^2 \to \mathbb{R}^2$ that fulfills two properties, namely

1. r is equivariant with respect to the set of transformations $\mathcal{T}$:

$$\forall T \in \mathcal{T}\ \forall A \in \mathcal{C}^2 :\ r(T(A)) = T(r(A))$$

2. r is a Lipschitz-continuous mapping in the following meaning:

$$\exists c \in \mathbb{R}_{>0}\ \forall A, B \in \mathcal{C}^2 : ||r(A) - r(B)|| \leq c\delta_H(A, B)$$

   In this context, c is called the quality of the reference point.

In [2] the Steiner point is shown to be a reference point of quality $\frac{4}{\pi}$. It is additionally shown, that this quality is optimal, which means that there cannot exist any reference point with a smaller Lipschitz constant. This is shown using strong functional-analytic tools and the axiom of choice. Therefore the proof is not constructive.

In [2] it is also shown how an approximation algorithm using reference points with approximation ratio $1 + c$ with respect to translations can be developed. For other sets of transformations, this algorithm can be used in a natural way to reduce several degrees of freedom.

Summarizing the lower bound on the quality of a reference point of $\frac{4}{\pi}$ and the upper bound of the algorithm using reference points of $1 + c$ with respect to translations, where c is the quality of any reference point, it seems reasonable that there are sets $A_1, A_2, ...$ which cannot be matched in a way that

$$\forall i \neq j \ : \ \delta_H(A_i + t_i, A_j + t_j) \leq (1 + \frac{4}{\pi} - \varepsilon) \cdot \delta_H^{opt}(A_i, A_j), \tag{1}$$

where $\delta_H^{opt}(A, B)$ is the optimal Hausdorff-Distance under translations, $\varepsilon \in \mathbb{R}_{>0}$ is any constant and $t_i \in \mathbb{R}^2$ are translation vectors. Observe that under these assumptions the vectors t_i can be interpreted as the reference points of the given sets.

In order to find compact convex sets in $\mathbb{R}^2$ which allow only an ε as small as possible in the above formula I have implemented two computer programs, both based on linear programming and AMPL. The first of these calculates the minimal Hausdorff-Distance of two figures, which can be achieved under translation. The second one determines the biggest ε and according translation vectors in a way that (1) is fulfilled for a given set of figures. Using the information given by the linear program solver, especially those about the tight inequalities, the program determines how to modify the given set of figures to get a new set, which only allows a smaller ϵ.

Using these two programs it was possible to find figures A_i so that $\varepsilon \approx 0.5$. Unfortunately the lower bound grows very slowly and it seems like the figures have to be very similar to induce a good bound. Therefore I would like to prove the following two conjectures:

Conjecture 1: Let $\delta_H^{opt}(A_i, A_j) > \alpha$ for all $i \neq j$ and some $\alpha > 0$. Then there are translation vectors t_i and a constant $\beta < 1 + \frac{4}{\pi}$ so that

$$\delta_H(A_i + t_i, A_j + t_j) \leq \beta \cdot \delta_H^{opt}(A_i, A_j).$$

(Dissimilar figures can be matched well.)

Conjecture 2: Let $A_1, ..., A_n$ be a set of figures inducing a lower bound β. Then there are figures $B_1, ..., B_n$ with $\delta_H^{opt}(B_i, B_j) < \delta_H^{opt}(A_i, A_j)$ inducing a lower bound $\beta' \geq \beta$.
(Figures inducing a good lower bound can be chosen very similar.)

Another problem we would like to solve is the extension to higher dimensions. Additionally, I am searching for a better algorithm for the calculation of the exact minimal Hausdorff-Distance that can be achieved under translations in arbitrary dimension.

While reading a report by Gerald Weber [3] my attention fell on a slightly stronger definition of regular reference points and some open problems connected to it. In this definition of regular reference points the set of allowed transformations to create the approximation is reduced. Therefore a general reference point needs not to be regular or it is regular but with a worse approximation ratio. However, up to now there is no general reference point known, that is not a regular reference point with the same approximation ratio. It may be interesting to find such a reference point.

Activities

Talks

- *Factoring numbers in O(log n) arithmetic steps*
  Noon Seminar of the TI-AG at FU Berlin on March 23

- *Topological Sweeplines and Horizon Trees*
  Noon Seminar of the TI-AG at FU Berlin on May 27

- *Lower bounds for shape matching with reference points*
  CGC-Colloquium at FU Berlin on May 10

Block Courses

- *Permutation Groups, Structures, and Polynomials*
  by Peter Cameron at Charles University in Prague,
  January 26 to March 2

- *Arrangements in Computational and Combinatorial Geometry* by Micha Sharir at Charles University in Prague, January 29 to February 27

Attended events

- *Monday Lectures and Colloquia* of CGC in Berlin
- *Noon Seminar* of the TI-AG at FU Berlin
- Lecture *Discrete Geometry: Polytopes and More* by Günther Ziegler at TU Berlin, April 20 to June 30
- *Berliner Algorithmen Tag* at TU Berlin on July 12

References

[1] H. Alt, B. Behrends, J. Blömer: 'Approximate matching of polygonal shapes', Proceedings 7th Annual Symposium on Computational Geometry, 1991, 186-193

[2] O. Aichholzer, H. Alt, G. Rote: 'Matching Shapes with a Reference Point', in International Journal of Computational Geometry and Applications, Volume 7, pages 349-363, August 1997

[3] G. Weber: 'The Centroid is a Reference Point for the Symmetric Difference in d Dimensions', Technical Report UoA-SE-2004-1, The University of Auckland, 2004