Semester Report SS 04 of Stephan Hell

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Results and Perspectives

In my third semester I was working in two different fields. First, I'm still in topological combinatorics where I was able to obtain a new result concerning the number of Tverberg partitions. Second, initiated by a project in the framework of the block course *Discrete Geometry (Polytopes and More)* of my advisor, I studied f–vector–type questions for cubical 4–polytopes. In the following, I will discuss both directions in detail. As usual, I suppose that the reader is familiar with my previous semester reports.

At the beginning of this semester, I have written up my new result on the number of Tverberg partitions and splitting necklaces. A Tverberg partition for a continuous map $f : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d$ is a collection of disjoint faces $F_1, \dots, F_q < \Delta^{(d+1)(q-1)}$ satisfying $\bigcap_1^q f(F_i) \neq \emptyset$. The existence of Tverberg partitions is proven for general d and prime powers q. The number of Tverberg partitions of such maps f is conjectured by G. Sierksma to be at least $((q-1)!)^d$. This problem is also known as the Dutch cheese problem. In [2], I was able to extend the currently best known lower bound for the prime case of [5] to the prime power case. The lower bound equals

$$\frac{1}{(q-1)!} \cdot \left(\frac{q}{r+1}\right)^{\lceil \frac{(d+1)(q-1)}{2} \rceil}, \text{ where } q = p^r \text{ and } p \text{ prime.}$$

The proof is inspired by the $\mathbb{Z}_p$–index version of the proof for the prime case in [4], and uses Volovikov's Lemma. The proof can also be seen as an application of Rade Živaljević's configuration space/test map paradigm. For big q and d my lower bound equals roughly the square root of the conjectured lower bound. See Table 1 for the state of art. Analogously, I have extended the lower bound for the number of splittings of generic necklaces from primes, see [5], to prime powers. There the lower bound equals

$$q \cdot \left(\frac{q}{r+1}\right)^{\lceil \frac{d(q-1)}{2} \rceil}.$$

My future studies will be based on Torsten Schöneborn's approach. He has

Lower bound \ q	prime	prime power	arbitrary
1 (existence)	Bárány et al. '81	Volovikov '96	open
[5]–type bound	[5]	✓	open
$((q-1)!)^d$ (Sierksma)	open	open	open

Table 1: Current state around the Topological Tverberg Theorem

reduced – among other results – the case $d = 2$ to a graph theoretic question. At the conference *Geometry, Topology, and Combinatorics*, KTH Stockholm, I had very encouraging discussions with Anders Björner and Rade Živaljević. Rade Živaljević has drawn my attention to the article [3] of Gil Kalai which I had overlooked until now.

Three–dimensional polytopes are quite well known combinatorial objects due to Steinitz (1906). The description of the set of all f–vectors $f(P) = (f_0, f_1, f_2)$ is complete, where P is a 3–polytope and f_i the number of its i–dimensional faces. In dimension 4 much less is known about the set of f–vectors. In my project, I studied f–vectors of the family of cubical 4–polytopes with few vertices. Recall, a polytope is called cubical if all its facets are combinatorial cubes. Gerd and Roswitha Blind have classified all cubical d–polytopes with up to $f_0 = 2^{d+1}$ vertices in series of five articles. The f–vector of a cubical 4–polytope is of the form $(f_0, f_0+2f_3, 3f_3, f_3)$. The following results on cubical 4–polytopes follow from their classification:

- The entry f_0 is even.
- Let P be a cubical 4–polytope, different from the 4–cube, then there is to every vertex v of P at least two diagonal vertices – two vertices that do not lie in facets through v.
- There are exactly ten cubical 4–polytopes with $f_0 \leq 32$ vertices and $f_0 \in \{16, 24, 28, 30, 32\}$. All of them can be obtained by a projection of the 5–cube to some 4–dimensional hyperplane.
- Let P be cubical 4–polytope and v a vertex of degree d. Then the following equality holds: $f_0(v) = 6d - 9$, where $f_0(v)$ is the number of vertices of P lying in facets through v.

The computer software *polymake* – developed in our group – is available for studying and visualizing polytopes. As part of my project, I have a obtained *electronic models* of the ten cubical 4–polytopes with ≤ 32 vertices: They can be visualized via their Schlegel diagrams. The main task was to study f–vectors of cubical 4–polytopes or, more generally, cubical 3–spheres on 34 vertices. Two constructions are known for constructing cubical 3–spheres out of cubical 3–spheres. The first one is called *(cubical) stacking*, and it adds 8 vertices to the polytope. The second one called *bistellar flip* construction adds 12 vertices. Neither of the constructions leads to a 4–polytope on 34 vertices, as there is no such polytope on 26 resp. 22 vertices (same for cubical 3–spheres). So $f_0 = 34$ is the last candidate for a gap. Using the lower bound theorem for general polytopes, and observing that the maximal degree of a cubical 4–polytope on 34 vertices equals 6, one obtains for $f_0 = 34$:

$$(34, 72, 57, 19) \leq (f_0, f_1, f_2, f_3) \leq (34, 102, 102, 34).$$

Using methods from [1], one can show that a 4–polytope on 34 vertices has a vertex of degree 6 with a stacked simplex as vertex figure. This proof involves shelling sequences and graph colorings of the 1–skeleton. The number of cases to be checked in the proof gets quite large and cannot be written up in a compact form. This method could be used for checking the classification of Blind and Blind, enumerating all cubical 3–spheres on a "small" number of vertices (especially the case $f_0 = 34$), and it has also shown up in other related problems. Axel Werner and I plan to start a computer project to implement the setup for this enumeration procedure.

Activities

- Lectures and Colloquia of the CGC
- Talk at Colloquium of the CGC, May 3, TU Berlin
- Talk on my block course project at *Mittagsseminar Diskrete Geometrie* at TU Berlin
- Attended Block course *Discrete Geometry (Polytopes and More)* of G. M. Ziegler in the framework of the Prague/Berlin DocCourse and Lectures *Diskrete Geometrie*
- Attended the Conference *Geometry, Topology, and Combinatorics*, July 3–6, 2004, KTH Stockholm, Sweden

Preview

- Annual workshop of the CGC, October 4–7, Stels, Switzerland
- Long stay for 4 months at KAM institute, Prague, visiting Prof. Dr. Jiří Matoušek, scheduled start in September 2004

References

[1] G. Blind and R. Blind, *Cubical 4–polytopes with few vertices*, Geom. Dedicata **66** (1997), pp. 223–231.

[2] S. Hell, *On the number of Tverberg partitions in the prime power case.* Preprint arXiv.math.CO/0404406, 2004. Submitted.

[3] G. Kalai, *Combinatorics with a geometric flavor: some examples*, in Visions in mathematics 2000 (GAFA Special Volume), Birkhäuser Basel, 2001, pp. 742–792.

[4] J. Matoušek, *Using the Borsuk–Ulam theorem*, Universitext, Springer–Verlag, Berlin, 2003. Lectures on topological methods in combinatorics and geometry.

[5] A. Vućić and R. T. Živaljević, *Notes on a conjecture of Sierksma*, Discr. Comput. Geom. **9** (1993), pp. 339–349.