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Field of Research:	Discrete and Combinatorial Geometry
Topic	Topology in DCG
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From January until April 2003 I spent a part of my "long stay" at ETH Zürich. In Zürich and back in Berlin I continued my work on topological combinatorics. One project was an upper bound on a topological lower bound on graph coloring using the so called box complex. After a result in the world of graphs the new goal is to generalize it to hypergraphs. My second field of work was to study the hom complex, a generalization of the box complex. In May 2003 I coorganized a conference in the mathematics and computer science section of the German foundation Cusanuswerk.

In 1978 László Lovász [4] proved Kneser's conjecture, a theorem of graph theory, using a classical topological ingredient, the Borsuk-Ulam theorem. Kneser's conjecture is concerned with the chromatic number of certain graphs. The Borsuk-Ulam theorem asserts that there is no continuous antipodal map from a sphere to its equator, i. e. $\nexists f : S^{d+1} \to S^d, f(-x) = -f(x)$.

Over the years Lovász' idea became the following (see [7]). Given a graph G, use all its complete bipartite subgraphs to define a simplicial complex $\mathcal{B}(G)$. In this so called *box complex* $\mathcal{B}(G)$ every simplex has a unique antipodal counterpart, i. e. there is a free $\mathbb{Z}_2$-action on $\mathcal{B}(G)$. On the sphere S^d we also have a $\mathbb{Z}_2$-action by the antipodal map. Hence we can define the *index* of the complex $\mathcal{B}(G)$ as the lowest dimension d of a sphere S^d into which we can map $\mathcal{B}(G)$ $\mathbb{Z}_2$-equivariantly:

$$\text{ind}(\mathcal{B}(G)) := \min\{d \mid \exists f : \mathcal{B}(G) \xrightarrow{\mathbb{Z}_2} S^d\}.$$

Based on the Borsuk-Ulam theorem Lovász proved the following essential relation between the chromatic number of a graph and the index of this complex (see [6]).

$$\chi(G) \geq \text{ind}(\mathcal{B}(G)) + 2.$$

This formula gives us a lower bound for the chromatic number of a graph. But this lower bound is not always tight. In joint work with Péter Csorba, Carsten Lange and Ingo Schurr we proved an upper bound to this lower bound. If a graph G does not contain a complete bipartite graph $K_{k,l}$ as a subgraph then the index of its box complex is bounded by $\text{ind}(\mathcal{B}(G)) \leq k + l - 3$.

Box complexes do not only exist for graphs. There are various analogue definitions for hypergraphs. With them one can state lower bounds on the chromatic number for hypergraph coloring, especially for coloring Kneser hypergraphs (see [5], [3], and [8]). Perhaps we can find an analogue version of our theorem and give upper bounds on these topological lower bounds.

László Lovász defined a generalization of the box complex and called it the hom complex. For any two graphs G and H the space $\text{Hom}(G, H)$ is a cell complex that represents all the graph homomorphisms from G to H (see [1]). For $G = K_2$ it is homotopy equivalent to the box complex.

The connectivity of the hom complex $\text{Hom}(G, H)$ for a family of graphs G has consequences for the chromatic number of the second graph H: Eric Babson and Dmitry Kozlov [1] proved the following conjecture by Lovász: If for any r the hom complex $\text{Hom}(C_{2r+1}, H)$ is k-connected then $\chi(H) \geq 4+k$. Graham Brightwell and Peter Winkler partly proved another conjecture by Lovász: If the hom complex $\text{Hom}(G, H)$ is connected for all graphs G of degree less than k then $\chi(H) > k$ (see [2]). The hom complex will be a field of my research in the next semester.

Activities

- Attendance of the Oberwolfach meeting on *Topological and Geometric Combinatorics* by Anders Björner, Gil Kalai, and Günter M. Ziegler, April 6 –12, Oberwolfach, Germany.

- Coorganized the Fachschaftstagung des Cusanuswerkes "Große Ideen der Mathematik und Informatik des 20. Jahrhunderts" May 28 – June 1, Uder, Germany.

- Attendance of the *Mittagsseminar* in Emo Welzl's research group *Theory of Combinatorial Algorithms* during my stay at ETH Zürich, including a talk on *The Hom Complex*, April 3, 2003.

- Attendance of the lecture *Polytope und Symmetrie* by Volker Kaibel, TU Berlin.

- Attendance of the lecture *Differentialtopologie* by Elmar Vogt, FU Berlin.

References

[1] Eric Babson and Dmitry N. Kozlov, *Topological obstructions to graph colorings.* arxiv:math.CO/0305300, 2003.

[2] Graham Brightwell and Peter Winkler, *Graph homomorphisms and long range action.* 2001.

[3] Igor Kríz, *Equivariant cohomology and lower bounds for chromatic numbers*, Trans. Am. Math. Soc., **333** (1992), pp. 567–577. correction ibid. 352, No.4, 1951-1952 (2000).

[4] László Lovász, *Kneser's conjecture, chromatic number, and homotopy*, Journal of Combinatorial Theory, Series A, **25** (1978), pp. 319–324.

[5] Noga Alon, Peter Frankl, and László Lovász, *The chromatic number of Kneser hypergraphs*, Trans. Am. Math. Soc, **298** (1986), pp. 359–370.

[6] Jiří Matoušek, *Using the Borsuk-Ulam theorem*, Universitext, Springer-Verlag, Berlin, 2003. Lectures on topological methods in combinatorics and geometry.

[7] Jiří Matoušek and Günter M. Ziegler, *Topological lower bounds for the chromatic number: A hierarchy.* preprint, August 2002.

[8] Günter M. Ziegler, *Generalized Kneser coloring theorems with combinatorial proofs*, Inventiones Mathematicae, **147** (2002), pp. 671–691.