Semester Report SS03 of Andreas Paffenholz

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Field of Research:	Discrete Geometry
Topic:	Flag Vectors of 4-Polytopes
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Field of Research and Results

In this semester I work on three different problems related to 4-polytopes.

Until the end of April I concentrated on finishing the paper "The E_t-Construction for Lattices, Spheres, and Polytopes" (joint with Günter M. Ziegler") [PZ03]. In this paper we present a new construction that can be applied to finite graded lattices to produce new lattices with some interesting properties. Namely, under certain conditions on the lattice we obtain k-simplicial and 2-simple lattices for some $k \geq 2$. If the lattice is a face lattice of a CW sphere we can moreover show that the resulting lattice is again the face lattice of a CW sphere. With some more elaborate geometric constructions we can also show that for some series of polytopes we can in fact find polytopes corresponding to the obtained lattices. With this we can in particular construct infinite series of rational 2-simple, 2-simplicial 4-polytopes and infinite series of 2-simple, $(d-2)$-simplicial d-polytopes.

In the following time I picked up some questions that were left open in our paper. Here in particular I worked on the question under which conditions we can expect the sphere we obtain by our construction to be realizable as a polytope. For this I tried to find some more general construction rules for polytopes obtained from the above rule, and I tried to describe realization spaces of the resulting polytopes.

The second problem I worked on in this semester was about cubical polytopes. Together with Carsten Lange I considered some questions about k-cubical and h-cocubcical d-polytopes. We were able to prove that there are no such polyotpes if $k + h \geq d + 1$. This is somehow similar to the case of k-simple, h-simplicial d-polytopes, were we only get the simplex if $k+h \geq d+1$. We have two different methods for the proof. One is based on topological arguments and shows that the universal cover of such a polytope must be infinite, the other one shows that one can equip such a polytope with a metric of nonpositive curvature, which implies that the universal cover is homeomorphic to Euclidean space. Unfortunately up to now we have no purely combinatrial proof for this fact.

In the following time we tried to extend our arguments to the case of 2-cubical, 2-cocubical 4-polytopes. But up to now we neither succeeded in extending the above proof to show that such polytopes do not exist nor did we succeed in constructing such an polytope. Though we can show that there cannot exist “easy” examples. That is, if there is such a polytope, it will have many facets and some of its facets must be rather complicated.

Some weeks ago I started working on questions related to Bier spheres. Originally, a Bier sphere is constructed from a Boolean lattice on the set $[n] := \{0, \dots, n\}$ and an ideal I in this lattice by the following rule:

$$\mathrm{Bier}(I) := \{(B, C) \mid \emptyset \subseteq B \subset C \subseteq [n], B \in I, C \notin I\}$$

I extended this definition to arbitrary face lattices of CW spheres and showed that there is a CW sphere having this face lattice. Both the construction and the proof of this are closely related to the above mentioned E_t-construction.

For the case of boolean lattices I also considered the question of shellability of the resulting (simplicial) spheres. Based on previous work of Anders Björner and Günter M. Ziegler I managed to prove that all these spheres are indeed shellable. The result though is not yet satisfying, as I can only prove the existence of a shelling but cannot give an explicit shelling order of the facets. Also, the original hope was to prove that the CW spheres for the construction applied to general face lattices of CW spheres (which are not neccessarily simplicial anymore) are all shellable. If this can be proven in analogy to the proof for Boolean lattices one first needs a more geometric interpretation of the shelling for the CW spheres from Boolean lattices.

Activities

- Block Course on “Network Algorithms” held by Prof. Möhring at TU Berlin, March, 31 – April, 11
- Attended the Lectures and Colloquia of the CGC
- Attended the Lecture on “Polytopes and Symmetry” by Volker Kaibel at TU Berlin
- Attended the Seminar on “Ricci Flow and the Geometrization Conjecture”, organized by Prof. K. Ecker at FU Berlin

- Attended the Oberseminar and the Brown-Bag-Seminar at TU Berlin
- Attended the Workshop on "Positive Scalar Curvature", February 16 – 22, List auf Sylt, organized by Prof. Bär, Prof. Ballmann, Prof. Leeb

Preview

I will attend the conference on "Polyhedral Surfaces" in St Petersburg, July 21 – 25.

References

[PZ03] A. Paffenholz and G. M. Ziegler *The E_t-construction for lattices, spheres, and polytopes*, `arXiv:math.MG/0304492`