Semester Report Summer '03 of Stephan Hell

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Field of Research:	Discrete Geometry
Topic:	Topological Methods in Combinatorics and Geometry
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This was my first semester in the CGC program. I was very glad about my start in the group *Discrete Geometry* of Günter Ziegler at TU Berlin.

Field of Research

My main interest is in the field of *Topological Methods in Combinatorics and Geometry.* Jiří Matoušek presents these methods in his recently published book [Mat03]. László Lovász's proof of *Kneser's Conjecture* in 1978 is often seen as the starting point of these methods. All currently known proofs of this conjecture use – implicitly or explicitly – the Borsuk–Ulam Theorem, which is standard for every book on *Algebraic Topology.*

In the following, I will shortly explain how these methods work: as a first step one reformulates the combinatorial resp. geometrical problem in topological terms, then one applies some topological result to solve the reformulated problem. More precisely, one defines some *configuration space* encoding all possible configurations of the problem, e. g. arrangements of points in $\mathbb{R}^d$, on the one hand, and some *test map* distinguishing configurations with some desired property from the others, e. g. by mapping the desired configurations to zero, on the other hand. This is what happens in $\mathbb{Z}_2$– or more generally in *G–Index Theory*, where G is some finite group reflecting some natural symmetry of the problem. This way one gets various non–trivial results about point configurations in $\mathbb{R}^d$, e. g. *on Radon's Theorem*, chromatic numbers, nonembeddability results for simplicial complexes, and more. My main focus is on the *Topological Tverberg Theorem*; more about this in the next section.

Recent work and perspectives

First I will discuss the *Topological Tverberg Theorem* and the current state of proofs, then I will go into some details, and I'll give possible future directions.

The *Tverberg Theorem* is a generalization of *Radon's Theorem* giving the minimal number $(d+1)(r-1)+1$ of points in $\mathbb{R}^d$ having a *Tverberg partition* into r sets. A *Tverberg partition* is a partition of the set of points into non–empty, disjoint subsets such that the intersection of their convex hulls is non–empty. This was proven directly without using any topological argument by Helge Tverberg in 1966 and [Tve81]; moreover it can be formulated in terms of an affine map from the $((d+1)(r-1)+1)$–dimensional simplex to $\mathbb{R}^d$. Replacing this affine map by a continuous map one gets the *Topological Tverberg Theorem*:

> Let $r \geq 2$, $d \geq 1$, and put $N := (d+1)(r-1)$. For every continuous map $f : \|\sigma^N\| \to \mathbb{R}^d$ there exist r disjoint faces $F_1, F_2, \ldots, F_r$ whose images under f intersect: $\bigcap_{i=1}^r f(\|F_i\|) \neq \emptyset$.

This theorem was first proven for the case when r is a prime number by Imre Bárány et al. in 1981. With the set–up of $\mathbb{Z}_r$–spaces and their index the proof is now rather short. Murat Özaydin was able to establish the theorem for prime powers r in [Öz87]; Karanbir Sarkaria has developed some very nice techniques for the prime case, and also proved it for prime powers. For arbitrary r the problem is still open. All known proofs for the prime power case use some deep results from algebraic topology about *characteristic classes*. Mark de Longueville gives in [dL99] the topological background needed for Sarkaria's work, and he explains the proof to a wider audience.

I do have regular working sessions with Torsten Schöneborn – diploma student in our group – who is studying the same subject. My focus for the arbitrary case is to concentrate on the "small" unproven cases, e. g. $r = 6$, $d = 2$. In my studies there are four main directions:

(i) Motivated by a conjecture of Gerard Sierksma on the lower bound for the number of Tverberg partitions, I have studied the question : How many Tverberg partitions (of a certain type) are there? I found some relation to Stirling numbers and other sequences.

(ii) I'm studying John Milnor's book *Characteristic classes* to get into the work of Sarkaria and Özaydin for the prime power case.

(iii) Following an idea of Torsten Schöneborn for the case $d = 2$, we study results on *crossing numbers* of the complete graph K_n and complete bipartite graphs $K_{n,n}$.

(iv) Starting with an article of B.J. Birch in 1959 I try to trace back the development of proofs, theorems and generalizations around the Tverberg theorem. Siniša Vrećica states a very general conjecture in [Vre03] and gives new proofs for new special cases. This conjecture implies or contains many classical results such as *Radon's theorem*, *Tverberg's Theorem*, *Rado's Theorem*, the *Ham Sandwich Theorem*, and non–embeddability results. It gives a very strong connection between geometrical and topological results.

Activities

- Lectures and Colloquia of the CGC
- Attended the Lectures of Volker Kaibel on "Polytopes and Symmetry"
- Attended the Oberseminar "Diskrete Geometrie" at TU Berlin

Preview

- CGC annual workshop, September 29 – October 1, 2003, Neustrelitz, Germany
- CGC Fall School on Computational Geometry, October 2 – 4, 2003, Neustrelitz, Germany

References

[dL99] M. de Longueville, *Notes on the topological Tverberg Theorem*, Discr. Math., 247 (2002), 271-297

[Mat03] J. Matoušek, *Using the Borsuk–Ulam Theorem*, Springer (2003)

[Öz87] M. Özaydin, *Equivariant maps for the symmetric group*, Preprint, University of Wisconsin–Madison, 1987, 17 pages

[Tve81] H. Tverberg, *A generalization of Radon's Theorem*, Bull. Austral. Math. Soc., 24 (1981), 321–325

[Vre03] S. T. Vrećica, *Tverberg's conjecture*, Discrete Comp. Geom., 29 (2003), 505–510