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Supervisor:	Prof. Dr. Günter M. Ziegler
Field of Research:	Discrete and Combinatorial Geometry
Topic	Topology in DCG
PhD Student	since June 1, 2001

Field of Research

This semester I worked in the field of topology and its applications to graph theory. I was able to find a shorter proof of a quantified version of the Borsuk-Ulam theorem. In joint work with Ingo Schurr and Carsten Lange I managed to show that graphs without a $K_{k,k}$-subgraph have a box complex of small index.

In 1978 Lovász [2] proved Kneser's conjecture, a theorem of graph theory, using a classical topological ingredient, the Borsuk-Ulam theorem. Kneser's conjecture is concerned with the chromatic number of certain graphs. The Borsuk-Ulam theorem asserts that there is no continuous antipodal map from a sphere to its equator, i. e. $\nexists f : S^{d+1} \to S^d, f(-x) = -f(x)$,

In the notation of [4], Lovász' idea can be interpreted as follows. Given a graph G, use all its complete bipartite subgraphs to define a simplicial complex $\mathcal{B}(G)$. In this so called *box complex* $\mathcal{B}(G)$ every simplex has a unique antipodal counterpart, i. e. there is a free $\mathbb{Z}_2$-action on $\mathcal{B}(G)$. On the sphere S^d we also have a $\mathbb{Z}_2$-action by the antipodal map. Hence we can define the *index* of the complex $\mathcal{B}(G)$ as the lowest dimension d of a sphere S^d into which we can map $\mathcal{B}(G)$ $\mathbb{Z}_2$-equivariantly (see [3]).

$$\operatorname{ind}(\mathcal{B}(G)) := \min\{d \mid \exists f : \mathcal{B}(G) \xrightarrow{\mathbb{Z}_2} S^d\}$$

Based on the Borsuk-Ulam theorem Lovász proved the following essential relation between the chromatic number of a graph and the index of this complex:

$$\chi(G) \geq \operatorname{ind}(\mathcal{B}(G)) + 2.$$

This formula gives us a lower bound for the chromatic number of a graph. But this lower bound is not always tight. In joint work with Ingo Schurr and Carsten Lange I succeeded in proving the following. If a graph G does not contain a complete bipartite graph $K_{k,k}$ as a subgraph then the index of its box complex is bounded by $\operatorname{ind}(\mathcal{B}(G)) \leq 2k - 3$.

To prove this we constructed a $\mathbb{Z}_2$-equivariant map from the box complex $\mathcal{B}(G)$ to a simplicial complex of dimension at most $2k-3$. Since the dimension of a space is an upper bound to its index this proves the assertion.

A negative application of this result is the chromatic number of the plane $\mathbb{R}^2$. The nodes of the corresponding graph are all points in the plane. Two of them are connected by an edge if their euclidean distance is 1. This graph does not contain a $K_{3,3}$-subgraph. Hence the corresponding index is at most 3. This shows that Lovász' bound could yield at most $\chi(\mathbb{R}^2) \geq 5$ but could not prove $\chi(\mathbb{R}^2) \geq 6$ or $\chi(\mathbb{R}^2) \geq 7$.

My second project was a quantification of the Borsuk-Ulam theorem. We can reformulate the Borsuk-Ulam theorem as "Every antipodal map $f : S^{d+1} \to S^d$ must be discontinuous." This discontinuity was quantified by Dubins and Schwarz [1] as follows. For any antipodal map $f : S^{d+1} \to S^d$ and any small $\varepsilon > 0$ there are two points x and y with distance less than ε such that their images are at least $|f(x) - f(y)| \geq \Delta(d)$ apart. This constant $\Delta(d)$ is the edge length of a regular simplex inscribed to the sphere S^d. In particular it does not depend on the map f. The proof by Dubins & Schwarz takes around 9 pages. I could shorten it to two pages. It is now based on the classical version of the Borsuk-Ulam theorem and some elementary linear algebra only.

Activities

- Organized the Fachschaftstagung des Cusanuswerkes "Diskrete Mathematik", May 8 – 12, 2002, Uder, Germany
- Conference in honour of Peter McMullen's 60th birthday, May 17 – 18, 2002, London
- CGC Spring School on "Approximation Algorithms for Hard Problems", May 20–23, 2002, Chorin, Germany
- "Mostly Discrete", A birthday colloquium for Martin Aigner, June 7, 2002, ZIB Berlin
- Poster Presentation at the review meeting of the CGC program by the DFG, June 24, 2002, FU Berlin
- Berliner Algorithmentag, July 19, 2002, FU Berlin

- Conference on "Discrete, Combinatorial and Computational Geometry", satellite conference of the ICM 2002, August 13 – 18, 2002, Beijing, China
- International Congress of Mathematicians, August 20 – 28, 2002, Beijing, China
- Lectures and Colloquia of the CGC
- Attended the lectures of Günter M. Ziegler on "Integer Programming" and of Volker Kaibel on "Enumerative Combinatorics" at TU Berlin
- Attended the Oberseminar Diskrete Geometrie and the Brown-Bag-Seminar at TU Berlin

References

[1] LESTER E. DUBINS AND GIDEON SCHWARZ, *Equidiscontinuity of Borsuk-Ulam functions*, Pacific J. Math., **95** (1981), pp. 51–59.

[2] LÁSZLÓ LOVÁSZ, *Kneser's conjecture, chromatic number, and homotopy*, Journal of Combinatorial Theory, Series A, **25** (1978), pp. 319–324.

[3] JIŘÍ MATOUŠEK, *Using the Borsuk-Ulam theorem (lectures on topological methods in combinatorics and geometry).* Book in preparation, written partially in collaboration with G. M. Ziegler and A. Björner, to be published by Springer.

[4] JIŘÍ MATOUŠEK AND GÜNTER M. ZIEGLER, *Topological lower bounds for the chromatic number: A hierarchy.* preprint, 2002.