Julian Pfeifle

Supervisor:	Prof. Günter M. Ziegler
Field of Research:	Discrete and Combinatorial Geometry
Topic:	Triangulated Complexes
PhD Student	at the program since March 1, 2000

Field of Research and Results

In this semester, I retook the subject of triangulated 3-spheres, and in joint work with Günter M. Ziegler was able to prove that asymptotically, there are far more simplicial 3-spheres than 4-polytopes!

As you may remember from my earlier semester reports, Kalai [7] proved in 1988 that for each $d \geq 4$, there are asymptotically many more triangulated d-dimensional spheres than $(d+1)$-dimensional polytopes. To briefly recapitulate, for any fixed dimension $d \geq 3$, Goodman and Pollack [2, 3] bounded the number of combinatorial types of simplicial $(d+1)$-polytopes on n vertices by $2^{O(n \log n)}$ (independently of the dimension!). The same asymptotic upper bound was later found to hold also for non-simplicial polytopes by Alon [1], and an asymptotically matching lower bound is provided by Shemer's construction of many neighborly polytopes [9]. The only known (and at first glance rather crude) upper bound $2^{O(n^{\lfloor (d+1)/2 \rfloor} \log n)}$ for the number of triangulated d-spheres on n vertices follows from Stanley's proof [10] of the upper bound theorem for spheres, which used the theory of Cohen-Macaulay rings. Surprisingly, Kalai [7] constructed in 1988 *"many triangulated spheres"*, namely $2^{\Omega(n^{\lfloor d/2 \rfloor})}$ of them. In particular, there are so many of them that *most*, in a very strong sense, must be non-polytopal.

However, Kalai's lower bound construction yields nothing for $d = 3$, as I was able to show in earlier work [8] that all triangulated 3-spheres he constructs are in fact polytopal. So the question concerning the relative number of 3-spheres and 4-polytopes remained unsettled: the boundary of every simplicial 4-polytope of course yields a 3-dimensional simplicial sphere, and Shemer's contruction [9] even yields $2^{\Omega(n \log n)}$ of them. (Note that by the Goodman-Pollack bounds on the number of combinatorial types of polytopes, this is even "optimal", up to a constant in the exponent(!).) In contrast, we know only few explicit constructions for non-polytopal triangulated 3-spheres, most of these based on some simple examples such as the Barnette sphere [4, 6].

Building on the techniques in [5], in joint work with GÜNTER M. ZIEGLER I was recently able to construct, for large n, far more triangulated 3-spheres on n vertices than there are 4-dimensional polytopes. Therefore, again by sheer number as in KALAI [7], *most* 3-spheres are not polytopal! The bound we have at the moment is $2^{\Omega(n^{5/4})}$ simplicial 3-spheres on n vertices, compared with at most $2^{20n \log n}$ simplicial 4-polytopes from the GOODMAN-POLLACK bound, and the upper bound of $2^{O(n^2 \log n)}$ 3-spheres given by the upper bound theorem for spheres.

The technique we use at the moment is to topologically decompose the 3-sphere into two solid handlebodies of large genus, several bipyramids over polygons, and many octahedra. The basic idea for this construction goes back to EPPSTEIN, and is accordingly called (a variant of) the *E-construction*. We triangulate the two handlebodies in some fixed way (using only few new vertices), do the same for the bipyramids, and then independently triangulate each of the many octahedra in two different ways. To be a little more specific, for some parameter p that can be arbitrarily large, the decomposition of the 3-sphere we obtain has $n = \Theta(p^4)$ vertices, $\Theta(p^4)$ bipyramids over $\Theta(p)$-gons, and $\Theta(p^5) = \Theta(n^{5/4})$ octahedra. Since octahedra are simplicial, we can triangulate their interiors independently in at least 2 ways each, and thus obtain $2^{\Omega(n^{5/4})}$ triangulated 3-spheres.

Activities

- Attendance of the lectures and colloquia of the CGC
- Presentation of the talk *Politopos de dimensión 4* at the IV Seminar on Discrete Mathematics, Universidad Politécnica de Madrid, May 30, 2002, Madrid
- Presentation of the talk *Long ascending paths on 4-polytopes* at the review meeting of the CGC program by the DFG, June 24, 2002, Berlin
- Presentation of the talk *Many triangulated 3-spheres*, Oberseminar TU Berlin, July 2, 2002
- Presentation of the talk *Many triangulated 3-spheres*, invited lecture of 30 minutes, on August 15, 2002, at *Discrete, Combinatorial, and Computational Geometry*, satellite conference of ICM 2002, August

13–19, 2002, organized by JACOB E. GOODMAN, RICHARD POLLACK, and CHUANMING ZONG, in Beijing, China

- *Conference in Honor of Peter McMullen's 60th birthday*, May 17–18, 2002, London
- CGC Spring School *Approximation Algorithms for Hard Problems*, May 20–23, 2002, in Chorin, Brandenburg
- *ACM Symposium on Computational Geometry SoCG 2002* June 5–7, 2002, in Barcelona
- *Berliner Algorithmentag BAT*, FU Berlin, July 19, 2002
- *International Congress of Mathematicians ICM 2002*, August 20–28, Beijing, China
- Attended the lectures *ADM III: Kombinatorische Optimierung* (ZIEGLER) and *Enumerative Kombinatorik* (KAIBEL), and the seminar *Semidefinite Programmierung* (ZIEGLER) at TU Berlin.
- Attended the seminars *Oberseminar Diskrete Geometrie*, and *Brown Bag Seminar* at TU Berlin

Preview

Two of the open questions that remain about triangulated 3-spheres are: Can we build even more of them? How many of them are shellable? I already have some partial results indicating a positive answer to the first question.

But the end of my CGC scholarship is approaching fast, so this semester I will also begin to write up my thesis. Rumor has it that this takes much more effort than expected ...

References

[1] NOGA ALON, *The number of polytopes, configurations and real matroids*, *Mathematika*, **33** (1986), 62–71.

[2] Jacob E. Goodman and Richard Pollack, *There are asymptotically far fewer polytopes than we thought*, Bull. Amer. Math. Soc., **14** (1986), 127–129.

[3] ——, *Upper bounds for configurations and polytopes in* $\mathbb{R}^d$, *Discrete Comput Geom*, **1** (1986), 219–227.

[4] David W. Barnette, *Diagrams and Schlegel diagrams*, in "Combinatorial structures and their applications", Proc. Calgary Internat. Conference 1969, New York, 1970, Gordon and Breach, 1–4.

[5] David Eppstein, Greg Kuperberg, and Günter M. Ziegler, *Fat* 4*-polytopes and fatter* 3*-spheres*. preprint `arXiv:math.CO/0204007`, 2002.

[6] Branko Grünbaum, *Diagrams and Schlegel diagrams*, *Notices Amer. Math. Soc*, **12** (1965), 578. Abstract 625-112.

[7] Gil Kalai, *Many triangulated spheres*, *Discrete Comput Geom*, **3** (1988), 1–14.

[8] Julian Pfeifle, *Kalai's squeezed* 3*-spheres are polytopal*, *Discrete Comput Geom*, **27** (2002), 395–407. DOI: 10.1007/s00454-001-0074-3.

[9] Ido Shemer, *Neighborly polytopes*, *Israel Journal of Mathematics*, **43** (1982), 291–314.

[10] Richard P. Stanley, *The upper-bound conjecture and Cohen-Macaulay rings*, *Stud. Appl. Math*, **54** (1975), 135–142.