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Supervisor:	Prof. Günter M. Ziegler
Field of Research:	Discrete and Combinatorial Geometry
Topic:	Flag Vectors of 4-Polytopes
PhD-Student:	at the program since April 2002

Field of Research and Results

This was my first semester as a graduate student in the CGC program. I worked on some questions related to the problem of finding a good characterization of all possible flag vectors $(f_S)_{S\subseteq\{0,\dots,3\}}$ of 4-polytopes, where f_S is the number of increasing chains of faces with one i-dimensional face for each $i \in S$. The flag vector has 15 entries, but due to the Euler and generalized Dehn-Sommerville equations only four of these numbers are independent. E.g. one could take (f_0, f_2, f_3, f_{03}). The convex hull of all possible flag vectors is a four-dimensional polyhedron F_4.

The goal is now to find inequalities in these four variables that describe the facet hyperplanes of F_4. But, so far, not much is known about this polyhedron (see [Bay87, Zie02] for a summary). This is in contrast to the case of 3-polytopes, where we have a complete characterization of all flag vectors due to Steinitz in 1906 [Ste06].

To improve the known bounds on the entries of the flag vector, it would be helpful to know more about 2-simple, 2-simplicial (2s2s) polytopes, in particular to have some infinite families of such polytopes (see [Bay87]). A 4-polytope is 2-simplicial, if all 2-faces are triangles, and it is 2-simple, if every edge is contained in exactly three facets. In Grünbaum's book [Grue67, p. 82, p. 170] it is stated that there are infinitely many 2s2s polytopes (Shepard resp. Perles and Shepard), but according to [EKZ02] both claims are incorrect.

In the same paper D. Eppstein, G. Kuperberg, and G. M. Ziegler [EKZ02] give a method for the construction of 2s2s 4-polytopes (the "E-construction"). They prove that the convex hull $E(Q)$ of the vertices of a simplicial 4-polytope Q having its edges tangent to the unit sphere together with the vertices of its polar dual Q^Δ is a 2s2s polytope. By a glueing construction from hyperbolic geometry they show that there are infinitely many simplicial edge-tangent polytopes Q and compute the flag vectors of $E(Q)$.

The resulting polytopes have a very special structure. All their facets are bipyramids over ridges of the dual simple polytope Q^Δ and the facets are all tangent to the sphere. Unfortunately most of these polytopes cannot be realized with rational coordinates and thus cannot be examined with `polymake` (cf. [GJ00]). Different from the 3-dimensional case, being edge-tangent is a severe restriction for a simplicial 4-polytope.

To get rid of the edge-tangency requirement in [EKZ02], I found a better view on the E-construction. Given a simple 4-polytope P, you can do the E-construction by taking as new facets certain positive linear combinations of any pair of neighbouring facet hyperplanes of P. The combinations are chosen such that all hyperplanes coming from a pair containing the same hyperplane of P meet in a single point.

Note that these are nonlinear conditions on the coefficients of the new inequalities. Thus it is difficult to establish the existence of linear combinations satisfying this additional requirement. In fact, it seems that the existence of a solution does depend on the specific geometric realization of the simple polytope P. Two months ago I succeeded in constructing an infinite family of rational 2s2s polytopes and give explicitly their inequality description. It has flag vector

$$(6 + 18n, 12 + 84n, 12 + 84n, 6 + 18n; 24 + 120n)$$

for any $n \in \mathbb{N}$. The underlaying simple polytopes are the duals of a stack of cross polytopes. Compared to the construction given in [EKZ02], my family of 2s2s polytopes is quite flexible, and it is different from all polytopes considered in [EKZ02] in that my polytopes are not tangent to the sphere. My construction does also apply to some other series of simple polytopes.

The second question I considered is the problem of finding bounds for the "fatness" parameter

$$F := \frac{f_1 + f_2 - 20}{f_0 + f_3 - 10}$$

associated to the f-vector of a polytope (except for the simplex). It is bounded from below by $\frac{5}{2}$ (see [Kal88, Zie02]). No upper bound is known, but it is conjectured that there is one. This is in contrast to the case of strongly regularly cellulated 3-spheres, which can have unbounded fatness (see [EKZ02]). At present, the 2s2s polytopes given in [EKZ02] are the fattest known 4-polytopes.

For cellulated 3-spheres it would also be interesting to establish (or disprove) the existence of a lower bound on their fatness. This is related to the problem of finding a sphere version of Kalai's lower bound theorem [Kal88] for polytopes. Kalai's proof for polytopes strongly uses the rigidity of such. But so far I have no results on this problem.

Activities

- CGC Spring School on "Approximation Algorithms for Hard Problems", May 20–23, 2002, Chorin, Germany
- Poster Presentation at the review meeting of the CGC program by the DFG, June 24, 2002, FU Berlin
- "Mostly Discrete", A birthday colloquium for Martin Aigner, June 7, 2002, ZIB Berlin
- Berliner Algorithmentag, July 19, 2002, FU Berlin
- Conference on "Discrete, Combinatorial and Computational Geometry", satellite conference of the ICM 2002, August 13 – 18, 2002, in Beijing, China
- International Congress of Mathematicians, August 20 – 28, 2002, in Beijing, China
- Lectures and Colloquia of the CGC
- Attended the lectures of Günter M. Ziegler on "Integer Programming" and of Volker Kaibel on "Enumerative Combinatorics" at TU Berlin
- Attended the Oberseminar Diskrete Geometrie and the Brown-Bag-Seminar at TU Berlin

Preview

- CGC annual workshop, October 10–12, 2002, Hiddensee, Germany

References

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