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Field of Research:	Complexity Theory and Geometry
Topic	Complex Tracing
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Field of Research

During the last months, I investigated the complexity of the *tracing problem* and the *reachability problem* in an algebraic setting. Originally both problems occurred in Dynamic Geometry (see [3]). Here, geometric constructions are represented by Geometric Straight-Line Programs (GSP). They consist of free points and dependent elements (like a line connecting two points, the intersection point of two lines, one of the two angular bisectors of two intersecting lines, one of the at most two intersection points of a line and a circle). An instance of a GSP is an assignment of fixed values to all free parameters and choices.

Since we work on dynamic geometry, we have to formalize movements of constructions. This is done via *continuous evaluations* ([3]): Given are continuous paths $p_i(t)$, $t \in [0, 1]$, of the free points $(i = 1, \ldots, k)$.

A continuous evaluation under the movement $\{p_i\}$ is an assignment of continuous paths o_j, $j = k+1, \ldots, m$, to all the dependent elements, such that for all $t \in [0, 1]$ the objects

$$(p_1(t), \ldots, p_k(t), o_{k+1}(t), \ldots, o_m(t))$$

form an admissible instance of the GSP.

There are two problems arising naturally from this setup:

- Reachability Problem:
  Given are two instances A and B of a GSP, where A is called starting instance and B final instance.
  Decide, whether there are paths $\{p_i\}$ of the free points, for which a continuous evaluation from A to B exists. In [3] is shown by a reduction of 3-SAT, that this problem is NP-hard.

- Tracing Problem:
  As in the reachability problem there are given a starting instance A and

a final instance B. Let p_A the position of the free points at instance A, and p_B their position at B. Furthermore a movement $\{p_i\}$ of the free points from p_A to p_B is given, for which there is a continuous evaluation. Decide whether a continuous evaluation given by the paths p_i and the starting instance A ends at B. In [3] is shown, again by reduction of 3-SAT, that this problem is NP-hard.

As mentioned in the beginning I have been working with GSPs in an algebraic setting (see [1]). Now the objects are elements of $\mathbb{C}$ with the operations $+, -, \cdot$ and $\sqrt{\ }$ instead of points, lines and circles with the geometric operations. In [1] you can find possibilities for translating a GSP in the algebraic world to one in the geometric world and vice versa.

The operations $+, -, \cdot$ are determined, which means that for each input there is exactly one output. In contrast to this, $\sqrt{\ }$ is not determined, since for each $z \in \mathbb{C}$, $z \neq 0$, we have two possible outputs (e.g. for $z = 4$ we have $\sqrt{4} = \pm 2$). So here $\sqrt{\ }$ plays the same rule as an angular bisector in the geometric world. Additionally 0 is a nonadmissible input for $\sqrt{\ }$, since $\sqrt{\ }$ is not analytic in 0.

If we have a (algebraic) GSP with just one free variable z, it corresponds to a Riemann surface (X, π). A continuous evaluation is a lifting of the path of the free variable z to the Riemann surface. So the reachability problem translates to the following question: Decide whether two points A and B lie in the same pathwise connected component of the surface (without nonadmissible points) described by the GSP.

The tracing problem can be described in the following way: Given are two points A and B on the Riemann surface X, where A is the starting point and B the final point. Furthermore a continuous path $p : [0,1] \to \mathbb{C}$ from $\pi(A)$ to $\pi(B)$ is given, which does not pass through nonadmissible points. Decide whether the lifting of p to the Riemann surface starting at A ends at B.

An implication of the NP-hardness of the tracing problem in the (real) geometric situation is that tracing is NP-hard in the algebraic setting, too. This is due to the fact that we can transform a tracing problem from the geometric setting into one of the algebraic setting using von Staudt constructions (see [1]).

In contrast to this, the complexity of the reachability problem is unknown in the algebraic setting. I found another proof for the NP-hardness of the tracing problem in the algebraic setting, which may lead to ideas for the algebraic reachability problem. It uses a switching mechanism based on the

expressions $\sqrt{\sqrt[2^k]{z}-1}$ for different ks.
The complex reachability problem might be useful for automatic theorem proving.

Activities

- Lectures and Colloquia of the graduate program, including the talk on June 16, 2002, titled *Complex Tracing*

- *Mittagsseminar Theoretische Informatik* at FU-Berlin, including the talks on
  April 4, 2002, titled *Complex Tracing* and on
  July 9, 2002, titled *Euler's Formula: A Topological Theorem*

- Spring School *Approximation Algorithms for Hard Problems*, May 20-23, 2002, Chorin, Germany

- Lecture *Kombinatorische Optimierung* by Günter Rote at FU-Berlin

- Berliner Algorithmen-Tag (July 19, 2002)

References

[1] U. Kortenkamp, *Foundations of Dynamic Geometry*, PhD-thesis, ETH Zürich, 1999

[2] U. Kortenkamp, J. Richter-Gebert, *Decision Complexity in Dynamic Geometry*, Proceedings of the ADG 2000, Springer Lecture Notes in Artificial Interlligence 2061, 2001

[3] J. Richter-Gebert, U. Kortenkamp, *Complexity Issues in Dynamic Geometry*, Proceedings of the Smale Fest 2000, 2001

[4] J. Richter-Gebert, *Grundlagen Geometrischer Operationen*, manuscript of a lecture, 2000

[5] W. Fischer, I. Lieb, *Ausgewhlte Kapitel aus der Funktionentheorie*, Vieweg 1988