Semester Report Dr. Deryk Osthus

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Field of Research:	Random discrete structures, probabilistic methods
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Results

Partitions of graphs with high minimum degree or connectivity

Hajnal [9] and Thomassen [12] independently proved that the vertex set of every highly connected graph G can be partitioned into sets S and T such that the graphs $G[S]$ and $G[T]$ induced by these sets still have high connectivity. With Daniela Kühn [10], I strengthened this result by showing that we can additionally require that every vertex in S has many neighbours in T:

For every ℓ there exists $k = k(\ell)$ such that the vertex set of every k-connected graph G can be partitioned into non-empty sets S and T such that both $G[S]$ and $G[T]$ are ℓ-connected and every vertex in S has at least ℓ neighbours in T.

This is best possible in the sense that we cannot additionally require the entire bipartite subgraph of G between S and T to have large minimum degree. Applying the result of Mader that every graph of sufficiently large average degree contains a subdivision of a given graph H to the graph $G[S]$ obtained by the above theorem, it immediately follows that every highly connected graph G contains a "non-separating" subdivision of H:

For every ℓ and every graph H there exists $k = k(\ell, H)$ such that every k-connected graph G contains a subdivision TH of H such that $G - V(TH)$ is ℓ-connected.

We also proved several related results about partitions of graphs.

Random planar graphs

Due to the high dependence between the existence of different edges in a random planar graph, rather little is known so far about the number of planar graphs and about their typical properties, in particular about their likely number of edges. Let $\mathcal{P}_n$ be the set of labelled planar graphs with n

vertices. Denise, Vasconcellos and Welsh [6] proved the upper bound $|\mathcal{P}_n| \leq n!\,75.8^{n+o(n)}$ and recently Bender, Gao and Wormald proved the lower bound $|\mathcal{P}_n| \geq n!\,26.1^{n+o(n)}$. Also, combining results by McDiarmid, Steger and Welsh and by Gerke and McDiarmid shows that almost all graphs in $\mathcal{P}_n$ have at least $13/7n$ edges. On the other hand, together with Hans Jürgen Prömel and Anusch Taraz [11], I showed the following upper bounds.

Almost all graphs in $\mathcal{P}_n$ have at most $2.56n$ edges. Also, $|\mathcal{P}_n| \leq n!\,37.3^{n+o(n)}$.

This considerably improves on our earlier bounds, which I described in the previous semester report. The proof relies on a result of Tutte on the number of plane triangulations, the above result of Bender, Gao and Wormald and the following result, which we also prove in [11]:

Every labelled planar graph G with n vertices and m edges is contained in at least $\varepsilon 3^{(3n-m)/2}$ labelled triangulations on n vertices, where ε is an absolute constant.

In other words, the number of triangulations of a planar graph is exponential in the number of additional edges which are needed to triangulate it. We also show that this bound on the number of triangulations is essentially best possible.

Popularity based random graph models

Although a great deal of research has been done on random graphs, only very recently has data on large real world networks become available. For instance, the world wide web may be viewed as a graph by considering the nodes or vertices to be the individual web-pages, while the connections or edges are links going from one document to another. Empirical studies of the world wide web and other large networks indicate structural characteristics which differ substantially from those of the graph models used in most previous research.

In particular, experimental evidence collected by several authors (see for example [4]) suggests that the fraction $P_{in}(d)$ of web pages with a particular number d of in-links decreases as a scale-free power law, namely proportional to $d^{-\gamma_{in}}$, with $\gamma_{in} \simeq 2.1$. This is also the case for networks as diverse as the movie actor collaboration graph ($\gamma \simeq 2.3$, [2]), scientific collaboration networks ($\gamma \simeq 3$, [2]) and the Western US Power Grid ($\gamma \simeq 4$, [2]).

This lead to the search for simple growth mechanisms which lead to the above (and other) observed phenomena. Dorogovtsev, Mendes and Samukhin [7] and Drinea, Enachescu and Mitzenmacher [8] introduced the following random graph model, which generalizes an earlier model of Barabási and Albert [1]: at each time step we add a new vertex incident to r edges. The other endpoints of these edges are chosen with probability proportional to their in-degrees plus an initial attractiveness ar, where a is a constant. Thus, popular vertices (i.e. those with large in-degree) will tend to become even more popular as time increases. For all a, $r \in \mathbb{N}$, we determine the asymptotic form of the degree distribution for most of the vertices. Confirming non-rigorous arguments in [7, 8], this shows that for such a, the proportion $P(d)$ of vertices of degree d almost surely obeys a power law, where $P(d)$ is of the form d^{-2-a} for large d. The case $a = 1$ (which corresponds to the model in [1]) was proved earlier by Bollobás, Riordan, Spencer and Tusnády [3].

Activities

- 26. Berliner Algorithmentag, ZIB, 16.2.2001 (with a talk on "Random planar graphs")
- Workshop on Random Structures, HU Berlin, 23.-24.3.2001
- Workshop on Combinatorics and Random Structures, HU Berlin, 5.-7.7.2001 (with a talk on "Induced subdivisions of graphs")
- Conference "Random Structures and Algorithms", Poznań, 6.-10.8.2001 (with a talk on "Induced subdivisions of graphs")
- Joint organization of the seminar "Algorithmen und Kombinatorik"
- Referee for the journals "Random Structures & Algorithms" and "SIAM journal on Discrete Mathematics"
- "Forschungsseminar Algorithmen und Komplexität" at the HU (with talks on "Colourings and independent sets in triangle-free graphs" and "Partitions of graphs of high minimum degree or connectivity")
- attendance of the lectures and the colloquium of the graduate school

References

[1] R. Albert and A.L. Barabási, *Emergence of scaling in random networks*, Science **286** (1999), 509-512.

[2] R. Albert, A.L. Barabási and H. Jeong, *Mean-field theory for scale-free random networks*, Physica A **272** (1999) 173-187, cond-mat/9907068.

[3] B. Bollobás, O. Riordan, J. Spencer and G. Tusnády, The degree sequence of a scale-free random graph process, *Random Struct. Alg.* **18** (2001), 279-290.

[4] A.Z. Broder, R. Kumar, F. Maghoul, P. Raghavan, S. Rajagopalan, R. Stata, A. Tomkins and J. Wiener, Graph structure in the web: experiments and models, *Proc. 9th WWW Conf.* (2000), 309-320.

[5] D. Osthus and G. Buckley, Popularity based random graph models leading to a scale-free degree distribution, *submitted*, 2001.

[6] A. Denise, M. Vasconcellos and D. Welsh, The random planar graph, *Congressus numerantium* **113** (1996), 61–79.

[7] S.N. Dorogovtsev, J.F.F. Mendes and A.N. Samukhin, Structure of growing networks: Exact solution of the Barabási-Albert model, cond-mat/0004434, 26 Apr 2000.

[8] E. Drinea, M. Enachescu and M. Mitzenmacher, Variations on random graph models for the web, technical report, Harvard University, Department of Computer Science, 2001.

[9] P. Hajnal, Partition of graphs with condition on the connectivity and minimum degree, *Combinatorica* **3** (1983), 95–99.

[10] D. Kühn and D. Osthus, Partitions of graphs with high minimum degree or connectivity, *submitted*, 2001.

[11] D. Osthus, H.J. Prömel and A. Taraz, On random planar graphs, the number of planar graphs and their triangulations, *submitted*, 2001.

[12] C. Thomassen, Graph decomposition with constraints on the connectivity and minimum degree, *J. Graph Theory* **7** (1983), 165–167.