Hard Problems in 3-Manifold Topology

Einstein Workshop on Discrete Geometry and Topology

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Embeddings in $\mathbb{R}^d$

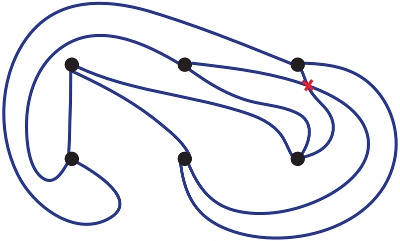
EMBED$_{k \to d}$

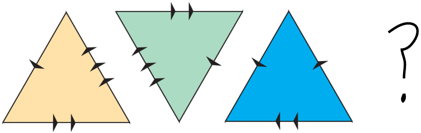
Problem: EMBED$_{k \to d}$

Given a k-dimensional simplicial complex, does it admit a piecewise linear embedding in $\mathbb{R}^d$?

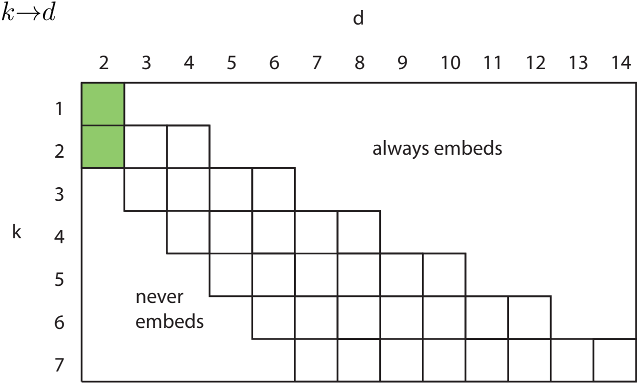
EMBED$_{1 \to 2}$ is Graph Planarity

EMBED$_{2 \to 3}$: does this 2-complex embed in $\mathbb{R}^3$?



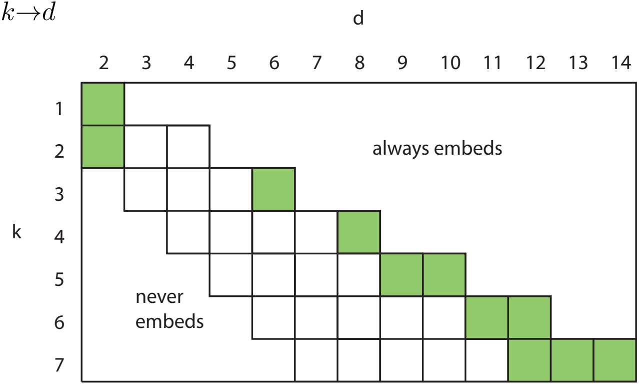


EMBED$_{k \to d}$



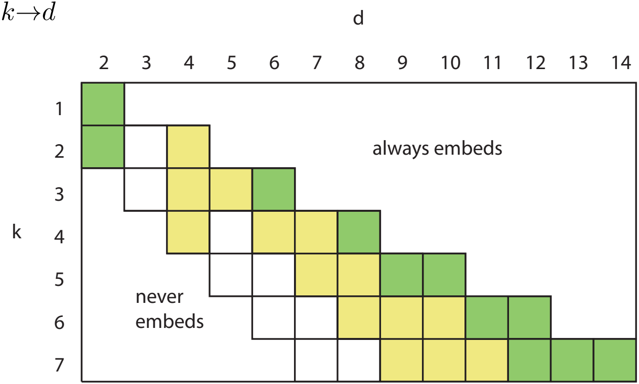
Polynomially decidable - Hopcroft, Tarjan 1971

EMBED$_{k \to d}$



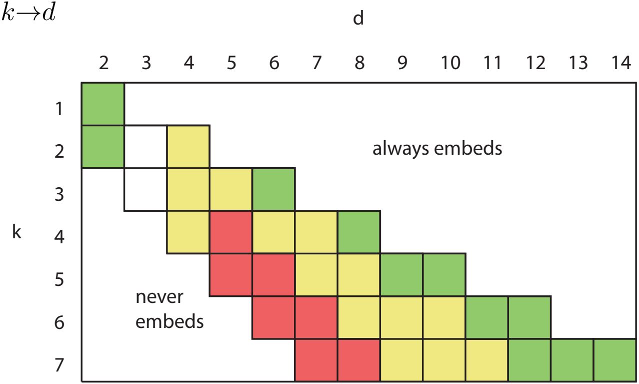
■ Polynomially decidable - Hopcroft, Tarjan 1971 ; Čadek, Krčál, Matoušek, Sergeraert, Vokřínek, Wagner 2013-2017

$\textsc{Embed}_{k \to d}$



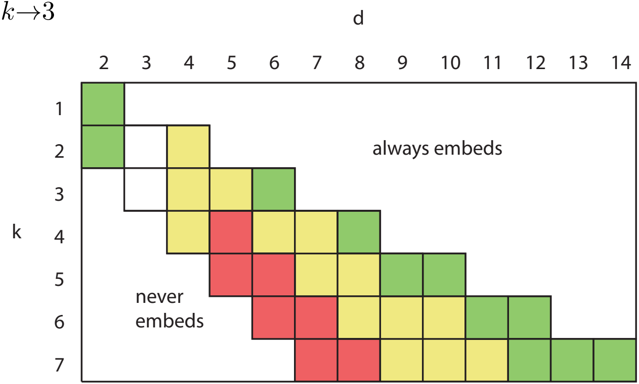
- Polynomially decidable - Hopcroft, Tarjan 1971 ; Čadek, Krčál, Matoušek, Sergeraert, Vokřínek, Wagner 2013-2017
- NP-hard - Matoušek, Tancer, Wagner ’11

EMBED$_{k \to d}$

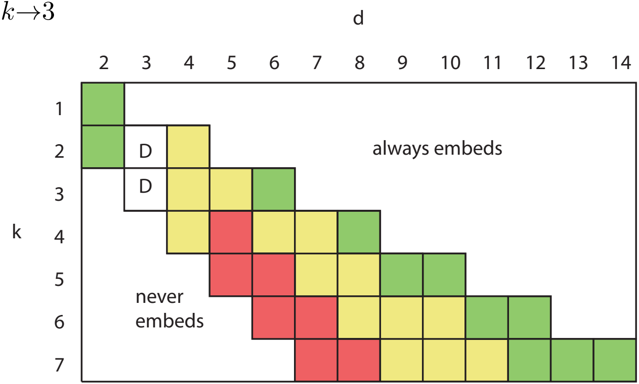


- Polynomially decidable - Hopcroft, Tarjan 1971 ; Čadek, Krčál, Matoušek, Sergeraert, Vokřínek, Wagner 2013-2017
- NP-hard - Matoušek, Tancer, Wagner '11
- Undecidable - Matoušek, Tancer, Wagner '11

$\text{EMBED}_{k\to 3}$



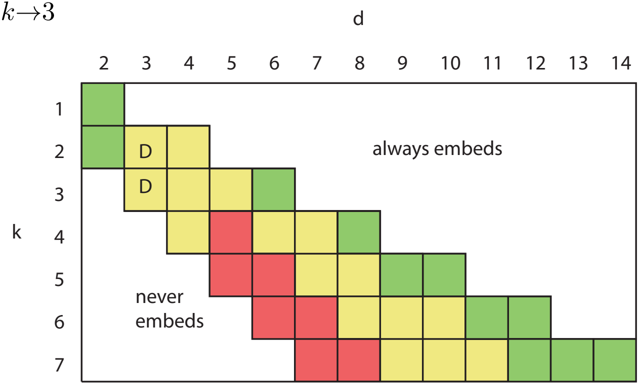
Embed$_{k\to 3}$



Theorem (Matoušek, S', Tancer, Wagner 2014)

*The following problems are **decidable**:*
Embed$_{2\to 3}$,
Embed$_{3\to 3}$, *and*
3-Manifold Embeds in S^3 (or $\mathbb{R}^3$).

$\textsc{Embed}_{k\to 3}$

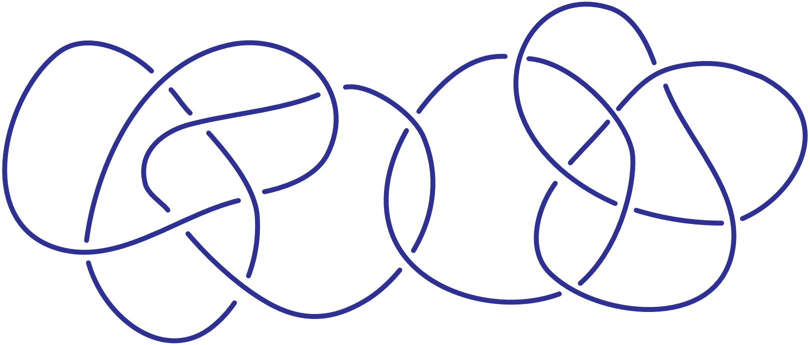


Theorem (de Mesmay, Rieck, S', Tancer 2017)

The following problems are ***NP-hard:***
$\textsc{Embed}_{2\to 3}$,
$\textsc{Embed}_{3\to 3}$, *and*
3-$\textsc{Manifold Embeds in}$ S^3 (or $\mathbb{R}^3$).

Knots and Links

A link diagram



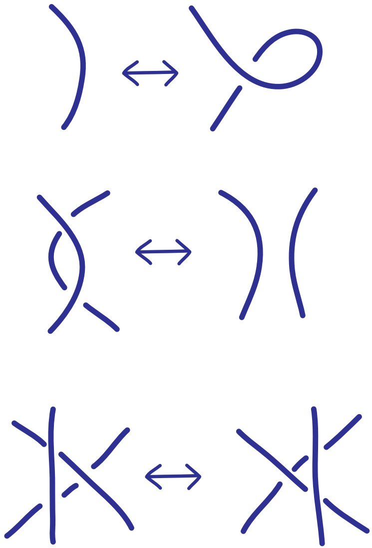
Reidemeister moves

Reidemeister (1927)

Any two diagrams of a link are related by a sequence of 3 moves (shown to the right).

Note:

Number of crossings may increase before it decreases.



Unlinking Number

Crossing Changes:

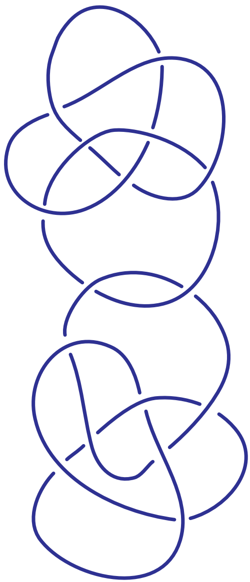
Any link diagram can be made into a diagram of an unlink (trivial) by changing some number of crossings.

Unlinking Number:

The minimum number of crossings *in some diagram* that need to be changed to produce an unlink.

Warning:

Minimum number may not be in the given diagram, so may need Reidemeister moves too.



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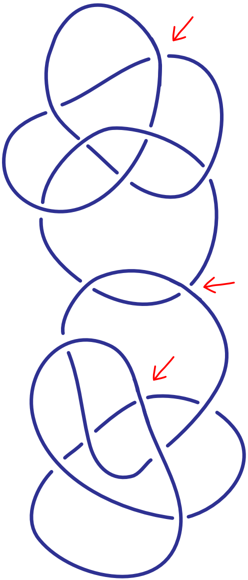


Given a link, 3 Questions:

Triviality

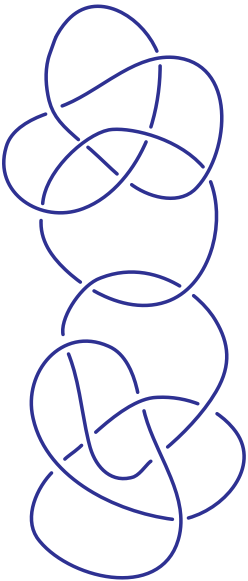
Is it trivial? Can Reidemeister moves produce a diagram with no crossings?

Trivial Sub-link

Does it have a trivial sub-link? How many components?

Unlinking Number

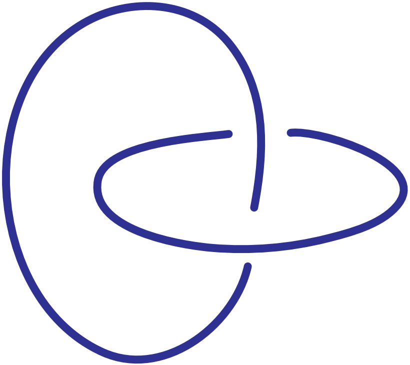
What is the unlinking number? How many crossing changes must be made to produce an unlink?



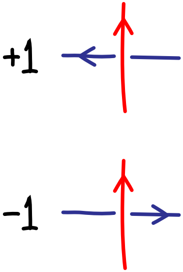
Hopf link

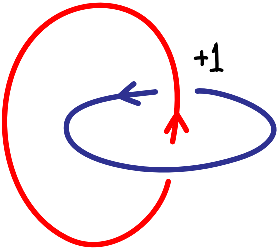
Triviality

Doesn't seem trivial, but how do you prove it?



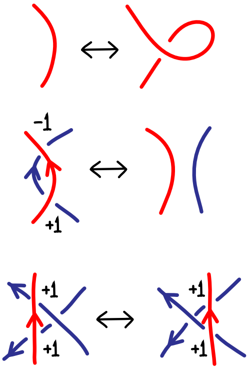
Linking number for two components:

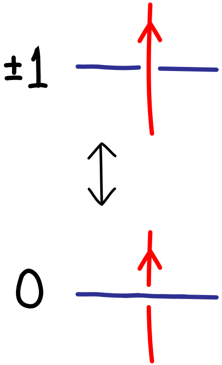




- choose red and blue and orient them
- for crossings of red over blue
- linking number is the sum of $+1$'s and -1's.

Linking number





Reidemeister moves

don't change the linking number!

A crossing change

changes the linking number by ± 1

Hopf Link

Triviality

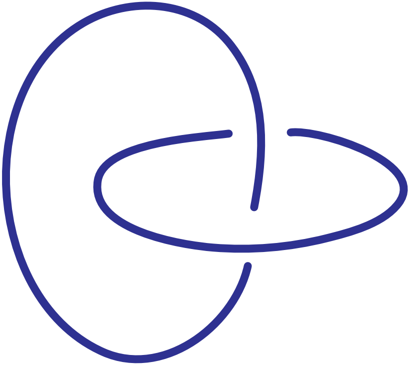
Not trivial. Linking number is not zero.

Trivial Sub-link

Maximal trivial sub-link has one component.

Unlinking Number

Unlinking number 1.



Borromean Rings

Triviality

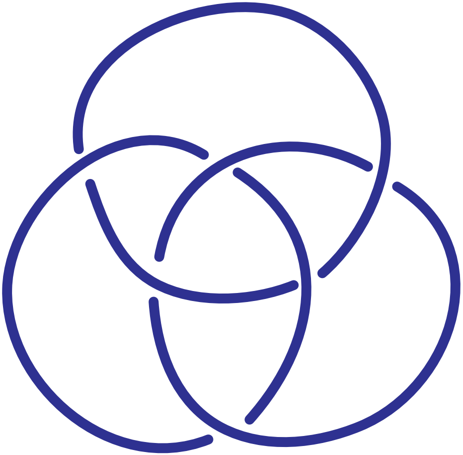
Not trivial. (But harder to prove, linking numbers are 0.)

Trivial Sub-link

Maximal trivial sub-link has **two** components.

Unlinking Number

Unlinking number 2. (Must show that it is greater than 1.)



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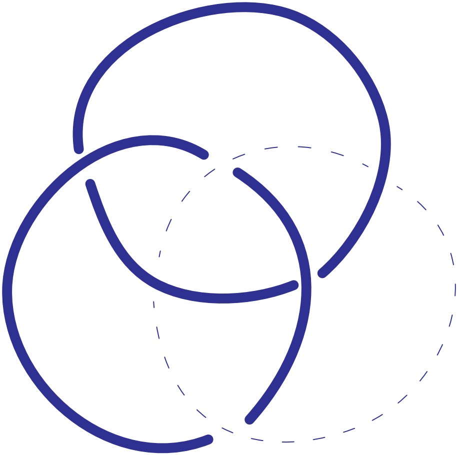


Whitehead Double of the Hopf Link

Triviality

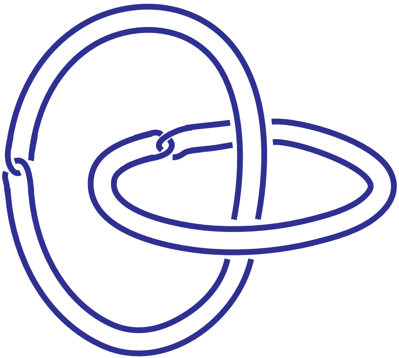
Not trivial. (Requires proof, linking numbers are 0.)

Trivial Sub-link

Maximal trivial sub-link has **one** component.

Unlinking Number

Unlinking number 1.



Whitehead Double of the Borromean Rings

Triviality

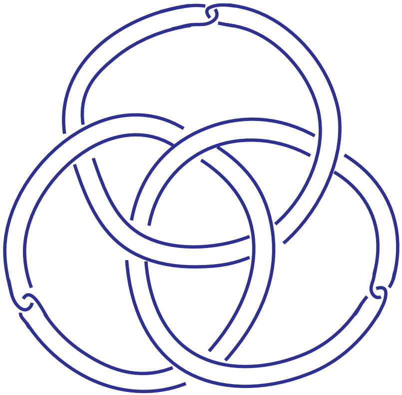
Not trivial. (Requires proof, linking numbers are 0.)

Trivial Sub-link

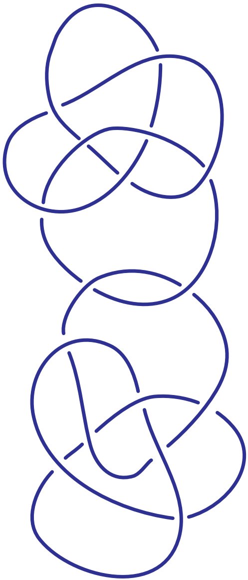
Maximal trivial sub-link has **two** components.

Unlinking Number

Unlinking number 1.



Decision Problems for Links



Triviality

Given a link diagram, does it represent a trivial link? (i.e., does it have a diagram with no crossings?)

Trivial Sub-link

Given a link diagram and a number n, does the link contain a trivial sub-link with n components?

Unlinking Number

Given a link diagram and a number n, can the link be made trivial by changing n crossings (in some diagram(s))?

What is known?

	NP	NP-hard
TRIVIALITY	✓	unlikely
TRIVIAL SUB-LINK	✓	✓
UNLINKING NUMBER	?	✓

TRIVIALITY & TRIVIAL SUB-LINK are in NP

Haken (1961); Hass, Lagarias, and Pippenger (1999)

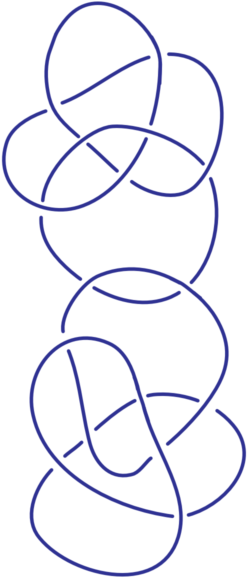
Unknot recognition is decidable [H], and, in NP [HLP].

Lackenby (2014)

For a diagram of an unlink, the number of moves required to eliminate all crossings is bounded polynomially in the number of crossings of starting diagram.

TRIVIAL SUB-LINK is also in NP

Apply this to the sub-diagram of the n component trivial sub-link.



TRIVIAL SUB-LINK is NP-hard

Problem: TRIVIAL SUB-LINK

Given a link diagram and a number n, does the link contain a trivial sub-link with n components?

Lackenby (2017)

(Non-trivial) SUB-LINK is NP-hard.

de Mesmay, Rieck, S' and Tancer (2017)

TRIVIAL SUB-LINK is NP-hard

Proof is a reduction from 3-SAT:

Given an (exact) 3-CNF formula Φ, there is a link L_Φ that has an n component trivial sub-link if and only if Φ is satisfiable. **(n = number of variables)**

TRIVIAL SUB-LINK is NP-hard

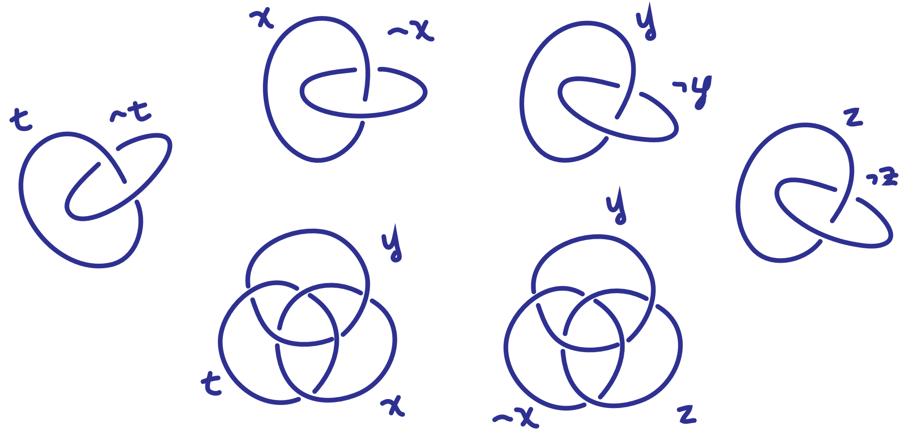
Constructing the link L_Φ :

$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \, x \ \vee \ y \ \vee \ z)$$

Given an (exact) 3-CNF formula, need to describe a link.

Constructing the link L_Φ :

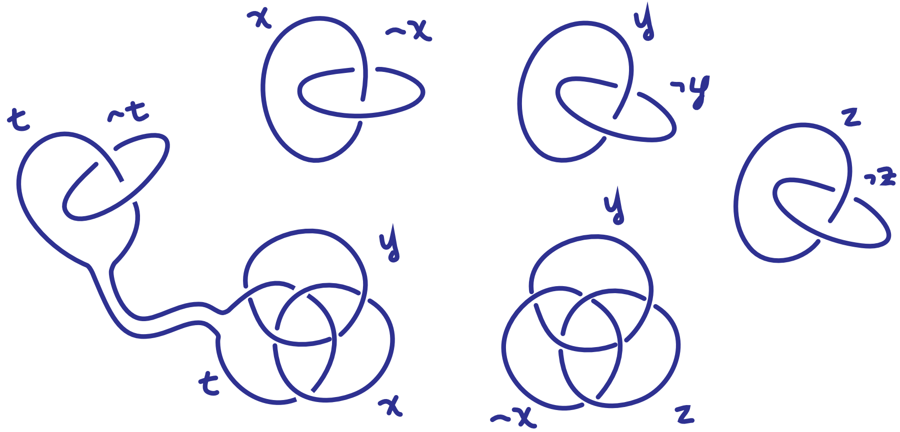
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \ x \ \vee \ y \ \vee \ z)$$



Draw **Hopf link** for each variable, **Borromean rings** for each clause.

Constructing the link L_Φ :

$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \ x \ \vee \ y \ \vee \ z)$$



Band each variable to its corresponding variable in the clauses.

Constructing the link L_Φ :

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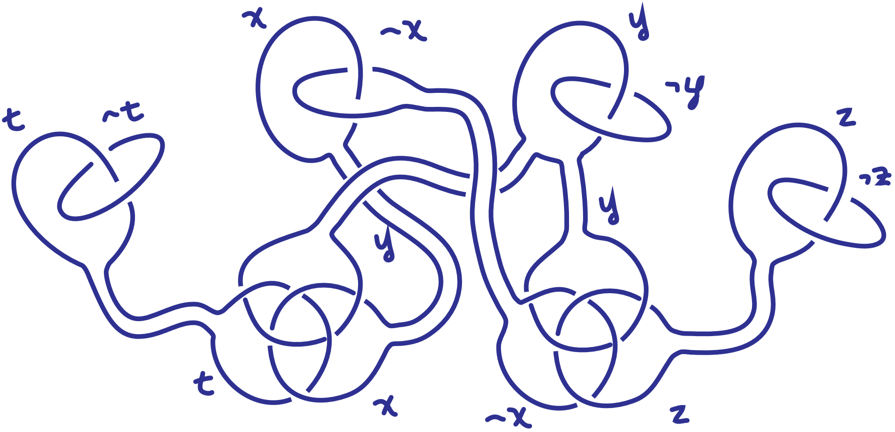

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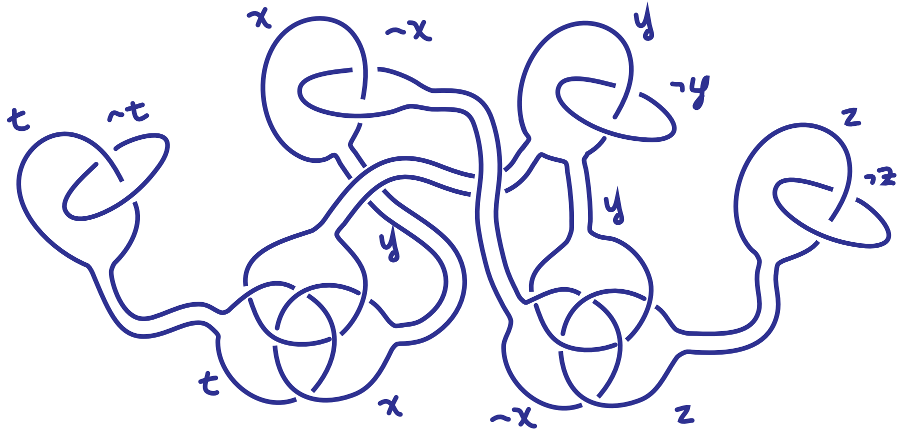
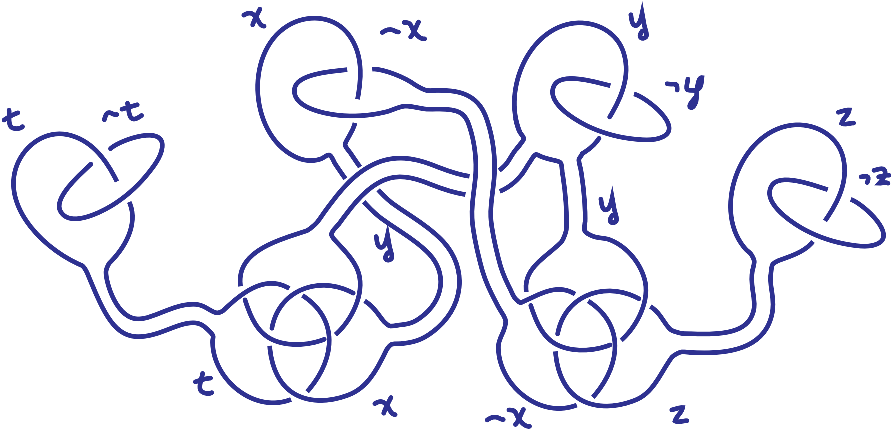


Each component is an unknot.

Φ satisfiable $\implies$ n component trival sub-link

Satisfiable $\implies$ n component trivial sub-link :

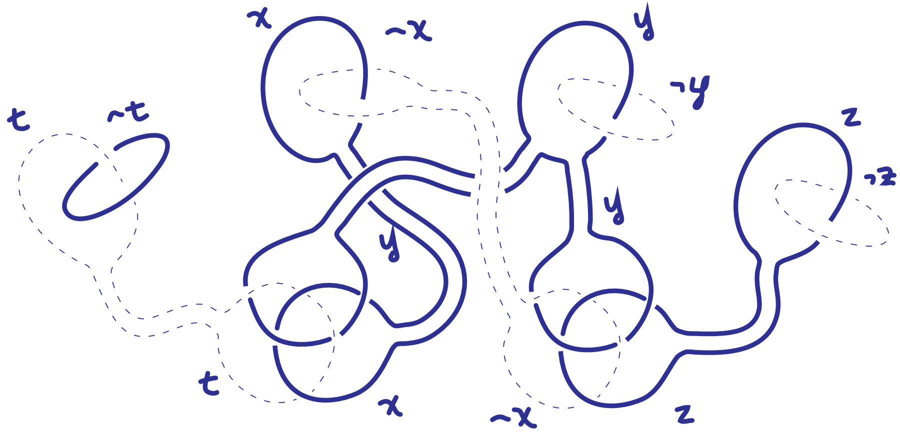
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg\, x \ \vee \ y \ \vee \ z)$$



Satisfiable: $t =$ TRUE; $x, y, z =$ FALSE.

Satisfiable $\implies$ n component trivial sub-link :

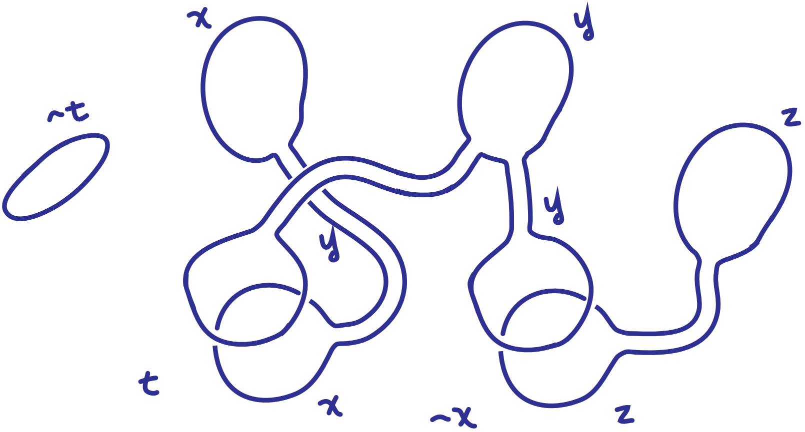
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



Erase TRUE components: $t, \neg x, \neg y, \neg z$.

Satisfiable $\implies$ n component trivial sub-link :

$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \ x \ \vee \ y \ \vee \ z)$$

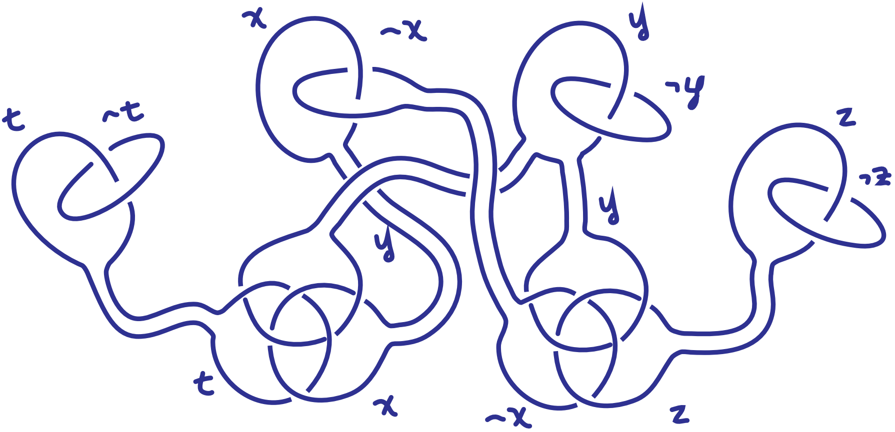


The FALSE components form an n component trivial sub-link.

n component trival sub-link $\implies$ Φ satisfiable

n component trivial sub-link $\implies$ satisfiable:

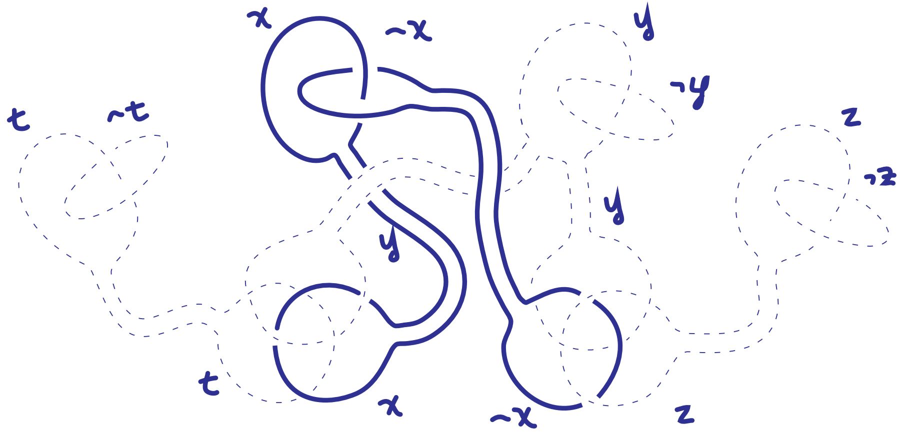
$$\Phi = (t \lor x \lor y) \land (\neg x \lor y \lor z)$$



Label the n trivial link components as FALSE, the others TRUE.

n component trivial sub-link $\implies$ satisfiable:

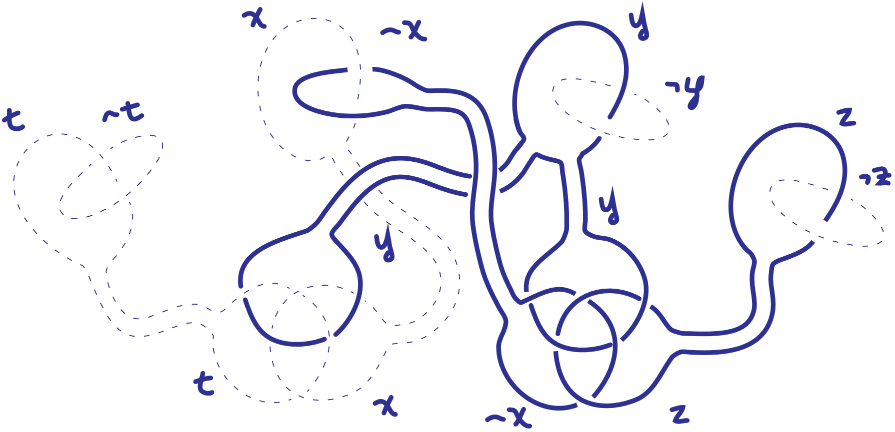
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \, x \ \vee \ y \ \vee \ z)$$



For each pair $(x, \neg x)$, one is TRUE the other FALSE.

n component trivial sub-link $\implies$ satisfiable:

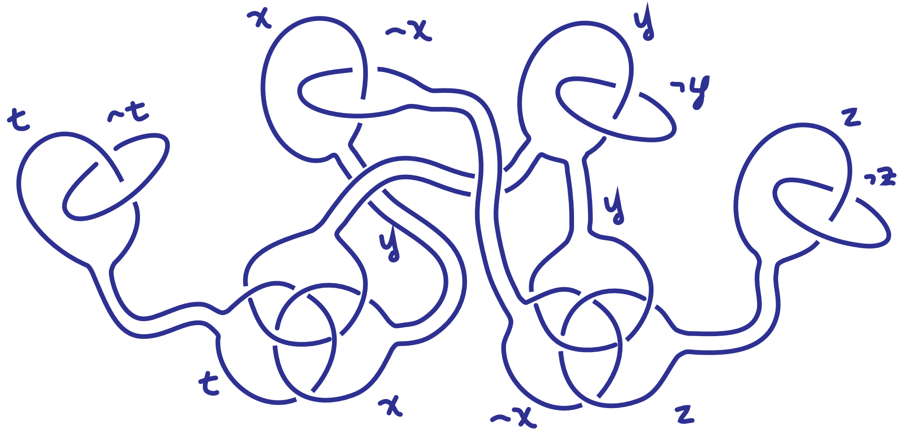
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



Each clause has a TRUE. (Borromean rings not sub-link of trivial link.)

n component trivial sub-link $\implies$ satisfiable:

$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$

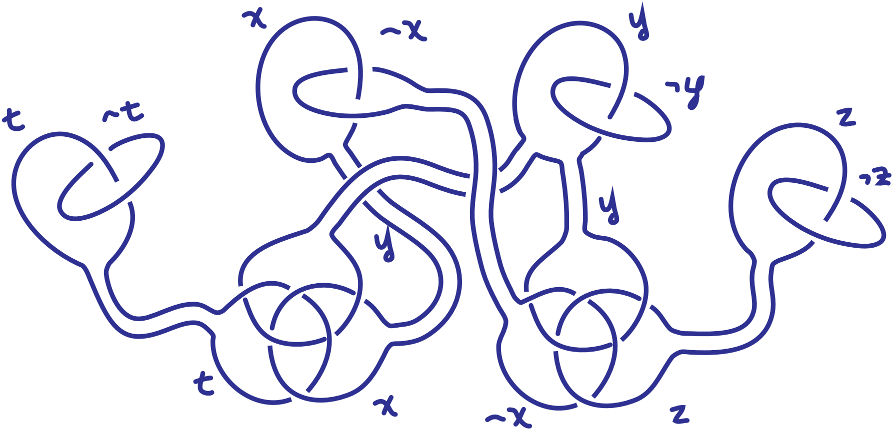


Therefore, Φ is satisfiable.

UNLINKING NUMBER is NP-hard

Unlinking Number is NP-hard

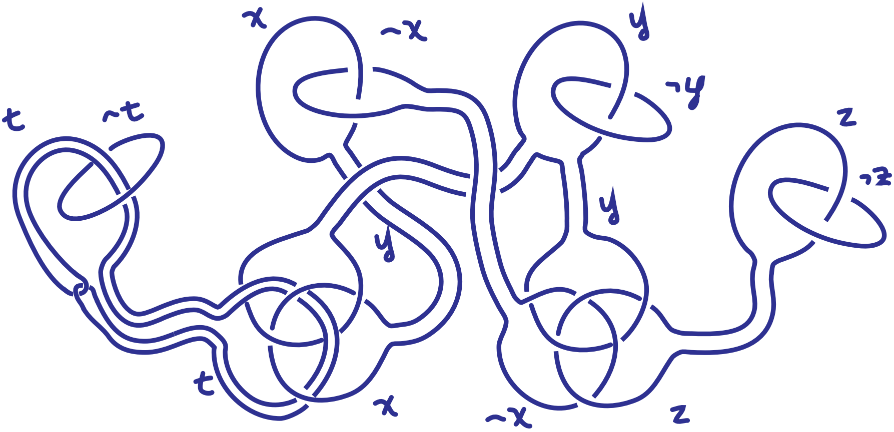
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \, x \ \vee \ y \ \vee \ z)$$



Related construction.

UNLINKING NUMBER is NP-hard

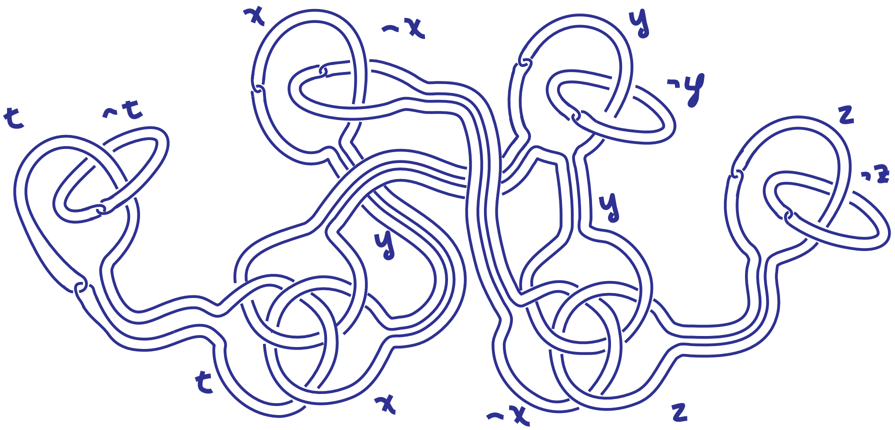
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \ x \ \vee \ y \ \vee \ z)$$



But replace each component with its Whitehead Double!

Unlinking Number is NP-hard

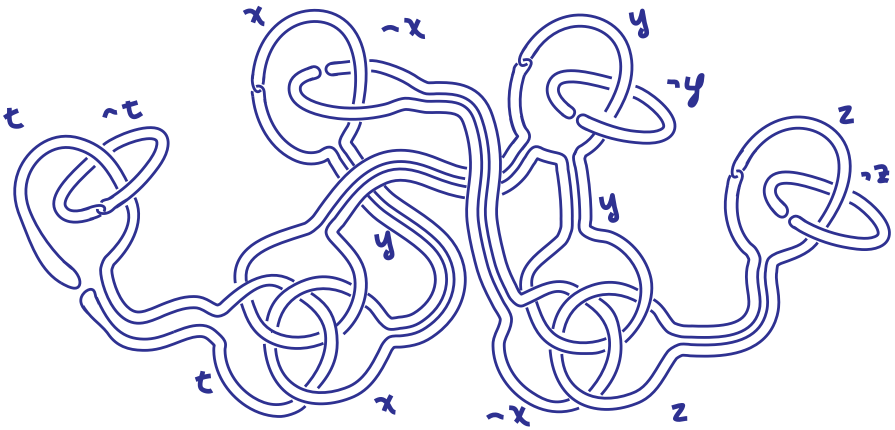
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



But replace each component with its Whitehead Double!

Φ satisfiable $\implies$ unlinking number n

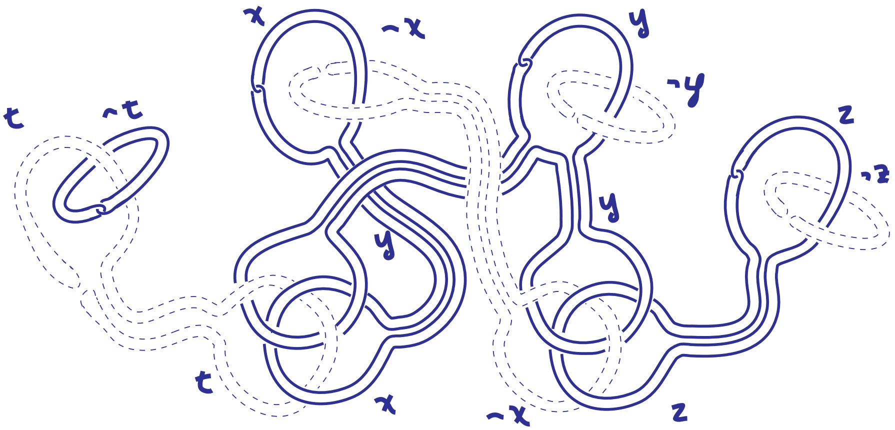
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



Φ is satisfiable, unclasp TRUE components.

Φ satisfiable $\implies$ unlinking number n

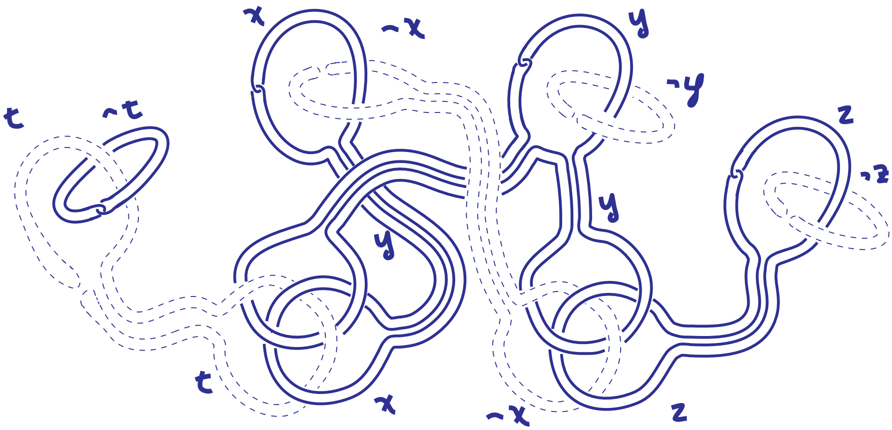
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



The TRUE components are an unlink, push to side.

Φ satisfiable $\implies$ unlinking number n

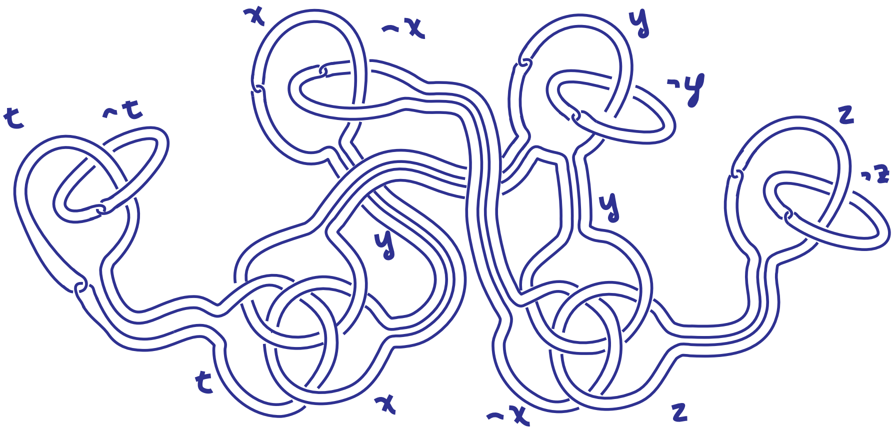
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg\, x \ \vee \ y \ \vee \ z)$$



What remains is also an unlink! $\implies$ unlinking number n.

unlinking number $n \implies \Phi$ satisfiable

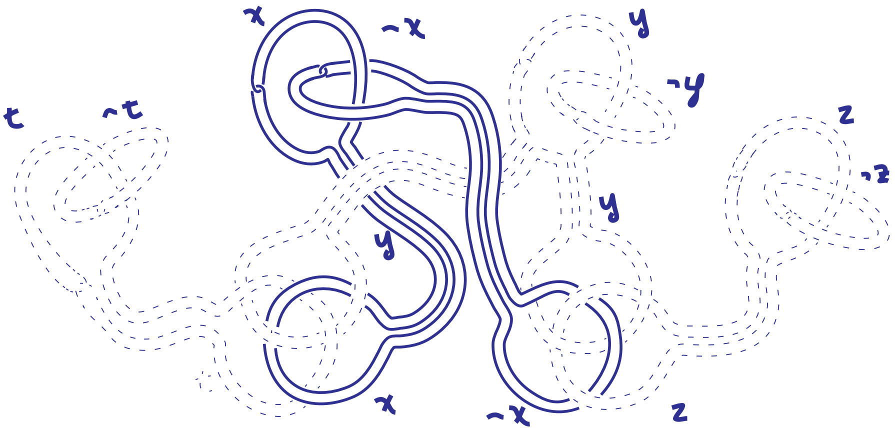
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



Unlinking number $n \implies$

unlinking number $n \implies \Phi$ satisfiable

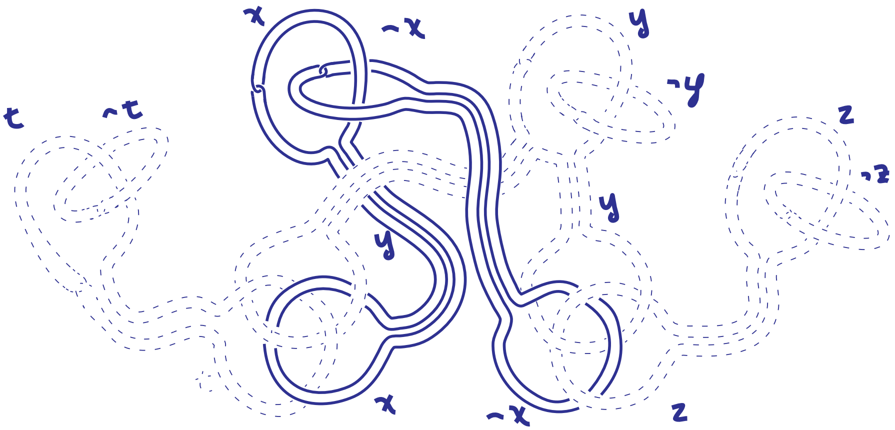
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



Unlinking number $n \implies$ each variable gets a crossing change.

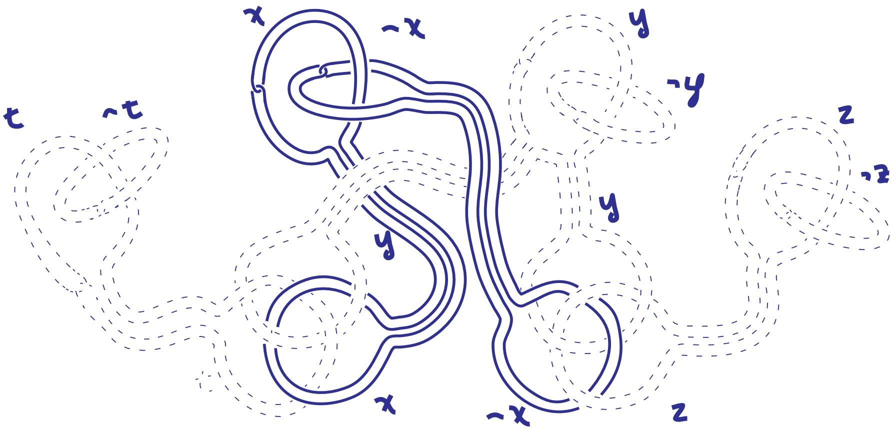
unlinking number $n \implies \Phi$ satisfiable

$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



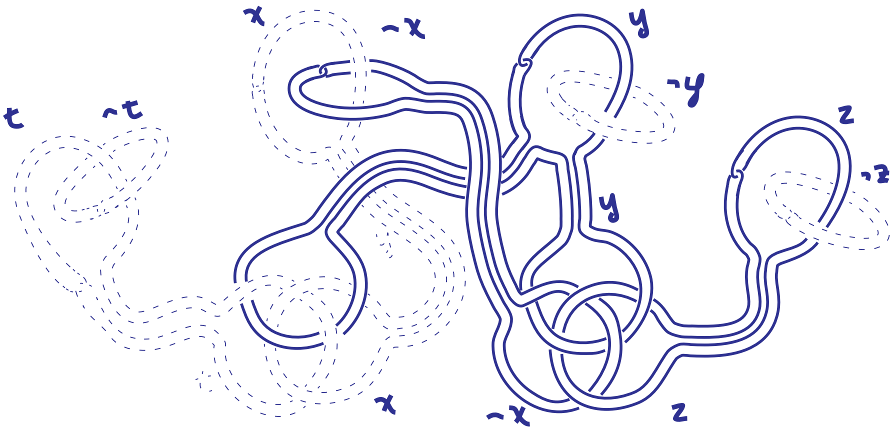
Crossing change affects **either** x or $\neg x$ (not both).

$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \, x \ \vee \ y \ \vee \ z)$$



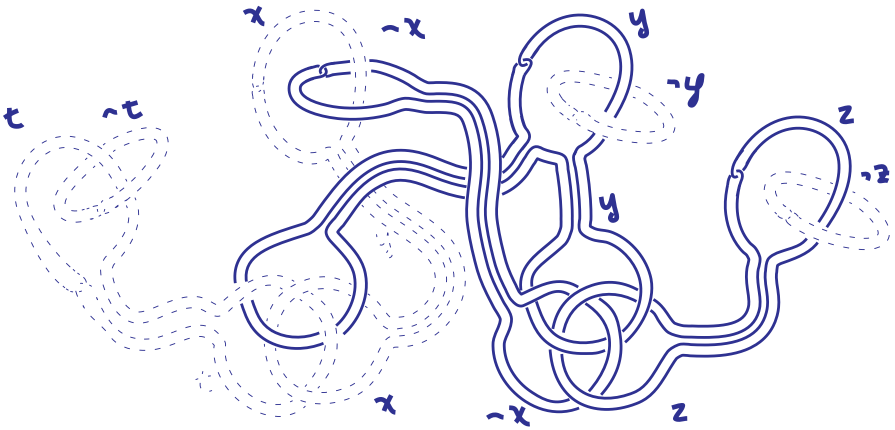
Call the changed components TRUE

$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg\, x \ \vee \ y \ \vee \ z)$$



Every Borromean clause has a changed crossing .

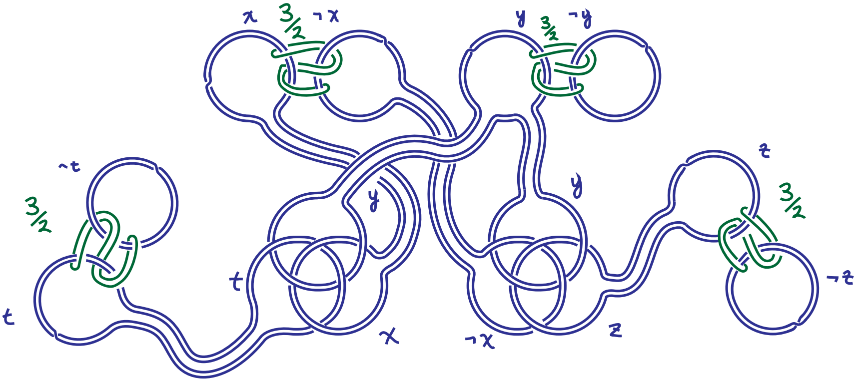
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



Every Borromean clause has a changed crossing $\implies$ Φ satisfiable.

$\textsc{Embed}_{2\to 3}$ is NP-hard

$\text{EMBED}_{2\to3}$ is NP-hard :



Uses a cabled link and **Dehn surgery**.

Open Questions:

	Knots	Links
TRIVIALITY	NP, co-NP[a]	NP
TRIVIAL SUB-LINK	n/a	NP-complete
UNLINKING NUMBER	?	NP-hard
3-MANIFOLD EMBEDS IN S^3	NP[b]	NP-hard

[a]Kuperberg; Lackenby; [b]Schleimer

Questions:

1. Is UNKNOTTING NUMBER, i.e., UNLINKING NUMBER for a single component, NP-hard?
2. Are UNLINKING NUMBER and $\text{EMBED}_{2\to3}$ in NP?

Thanks!