Three questions on graphs of polytopes

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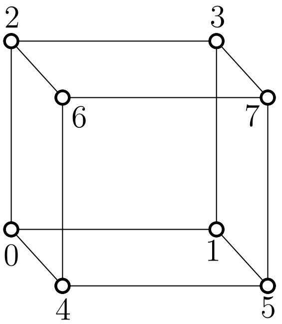
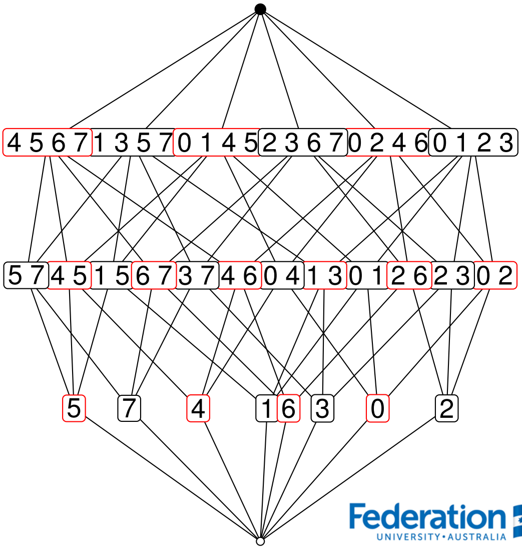
Outline

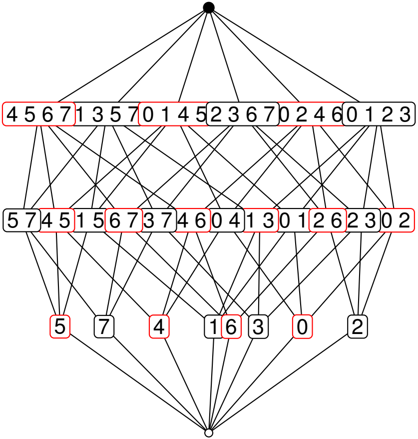
A polytope as a combinatorial object





Reconstruction of polytopes (Dolittle, Nevo, Ugon & Yost)

- The *k*-skeleton of a polytope is the set of all its faces of dimension $\leq k$.
- *k*-skeleton reconstruction: Given the *k*-skeleton of a polytope, can the face lattice of the polytope be completed?



Some known results

- (Grünbaum '67) Every d-polytope is reconstructible from its $(d-2)$-skeleton.

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- For $d \geq 4$ there are pairs of d-polytopes with isomorphic $(d-3)$-skeleta:
  - a bipyramid over a $(d-1)$-simplex and,
  - a pyramid over the $(d-1)$-bipyramid over a $(d-2)$-simplex.

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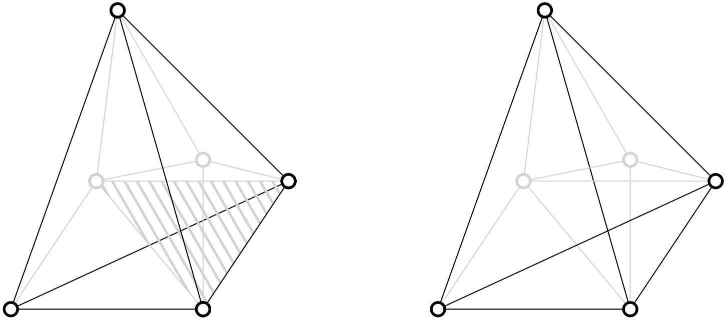
Polytopes nonreconstructible from their graphs



(a) pyr(bipyr(T_2))   (b) bipyr(T_3)

Some known results

- (Blind & Mani, '87; Kalai, '88) A simple polytope is reconstructible from its graph.

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- Call d-polytope $(d-k)$-simple if every $(k-1)$-face is contained in exactly $d-k-1$ facets.
- A simple d-polytope is $(d-1)$-simple.
- (Kalai, '88) A $(d-k)$-simple d-polytope is reconstructible from its k-skeleton.

Call a vertex of a d-polytope nonsimple if the number of edges incident to it is $> d$.

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Theorem (Doolittle-Nevo-PV-Ugon-Yost, '17)

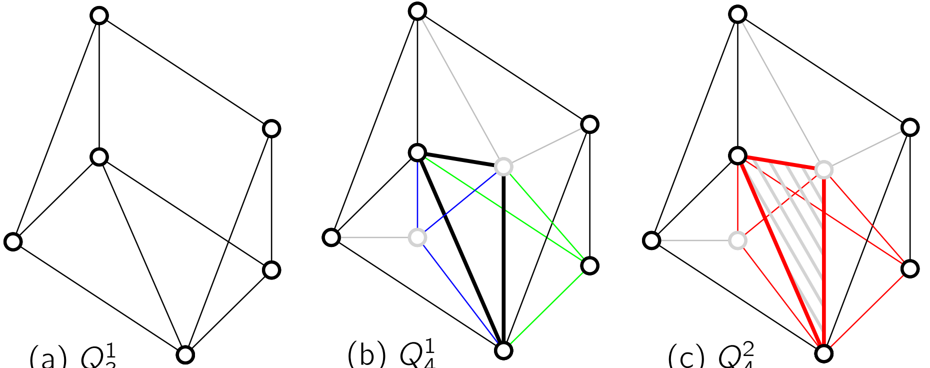
Let P be a d-polytope. Then the following statements hold.

1. *The face lattice of any d-polytope with at most two nonsimple vertices is determined by its graph (1-skeleton);*
2. *the face lattice of any d-polytope with at most $d-2$ nonsimple vertices is determined by its 2-skeleton; and*
3. *for any $d > 3$ there are two d-polytopes with $d-1$ nonsimple vertices, isomorphic $(d-3)$-skeleton and nonisomorphic face lattices.*

The result (1) is best possible for 4-polytopes.

Nonisomorphic 4-polytopes with 3 nonsimple vertices

- Construct a d-polytope Q_1^d.
- The polytope Q_2^d is created by "**gluing**" two simplex facets of Q_1^d along a common ridge to create a bipyramid of Q_2^d.



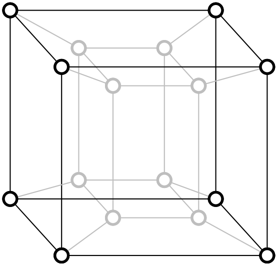
(a) Q_3^1 (b) Q_4^1 (c) Q_4^2

Open problem

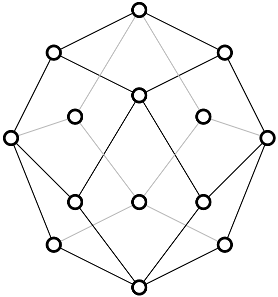
Problem

Is every d-polytope with at most $d - 2$ nonsimple vertices reconstructible from its graph?

A cubical d-polytope is a d-polytope in which every facet is a $(d-1)$-cube.



(a) 4-cube



(b) cubical 3-polytope

When referring to graph-theoretical properties of a polytope such as minimum degree and connectivity, we mean properties of the graph $G = (V, E)$ of the polytope.

- (Balinski '61) The graph of a d-polytope is d-(vertex)-connected.

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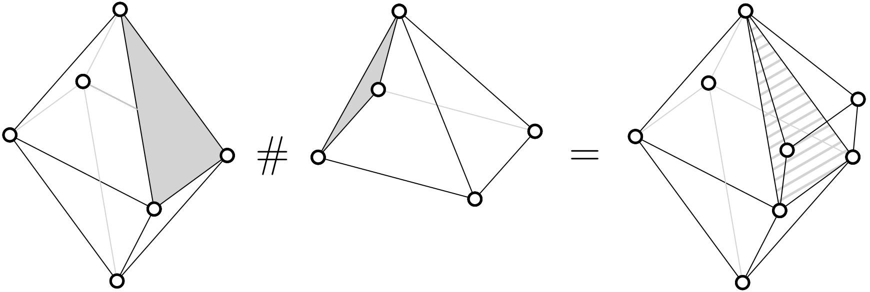
- (Perles & Prabhu '93) Removing the subgraph of a k-face from the graph of a d-polytope leaves a $\max(1, d-k-1)$-connected subgraph.

Minimum degree vs connectivity



(a) P_1 (b) P_2 (b) $P_1\#P_2$

Figure: There are *d*-polytopes with high minimum degree which are not $(d+1)$-connected.

Theorem (Connectivity Theorem; Hoa, PV & Ugon)

Let $0 \leq \alpha \leq d-3$ and let P be a cubical d-polytope with minimum degree at least $d+\alpha$. Then P is $(d+\alpha)$-connected.

Furthermore, if the minimum degree of P is exactly $d+\alpha$, then, for any $d \geq 4$ and any $0 \leq \alpha \leq d-3$, every separator of cardinality $d+\alpha$ consists of all the neighbours of some vertex and breaks the polytope into exactly two components.

This is best possible in the sense that for $d=3$ there are cubical d-polytopes with minimum separators not consisting of the neighbours of some vertex.

Connectivity Theorem and $d = 3$

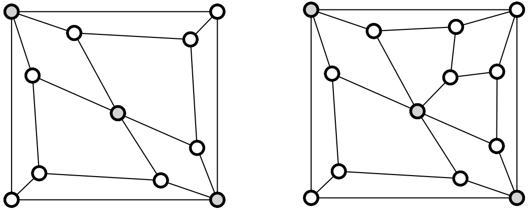
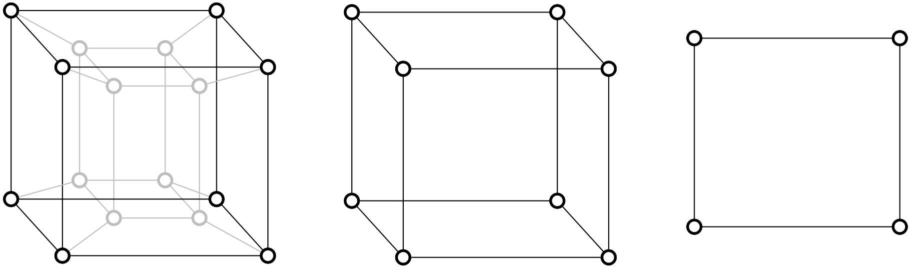


Figure: Cubical 3-polytopes with minimum separators not consisting of the neighbours of some vertex. Vertex separator coloured in gray.

Note: Infinitely many more examples can be generated by using well know expansion operations such as those in “Generation of simple quadrangulations of the sphere” by Brinkmann et al.

Connectivity Theorem and cubes



(a) 4-cube (b) 3-cube (c) 2-cube

Figure: Every minimum separator of a cube consists of the neighbours of some vertex.

Note: This can be proved by induction on d, considering the effect of the separator on a pair of opposite facets.

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Ingredient 1: Strongly connected $(d-1)$-complex. A finite nonempty collection $\mathcal{C}$ of polytopes (called faces of $\mathcal{C}$) satisfying the following.

- The faces of each polytope in $\mathcal{C}$ all belong to $\mathcal{C}$, and
- polytopes of $\mathcal{C}$ intersect only at faces, and
- each of the faces of $\mathcal{C}$ is contained in $(d-1)$-face, and
- for every pair of facets F and F', there is a path $F = F_1 \cdots F_n = F'$ of facets in $\mathcal{C}$ such that $F_i \cap F_{i+1}$ is a $(d-2)$-face, ridge, of $\mathcal{C}$.

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(Sallee '67) The graph of a strongly connected $(d-1)$-complex is $(d-1)$-connected.

Examples of strongly connected $(d-1)$-complexes

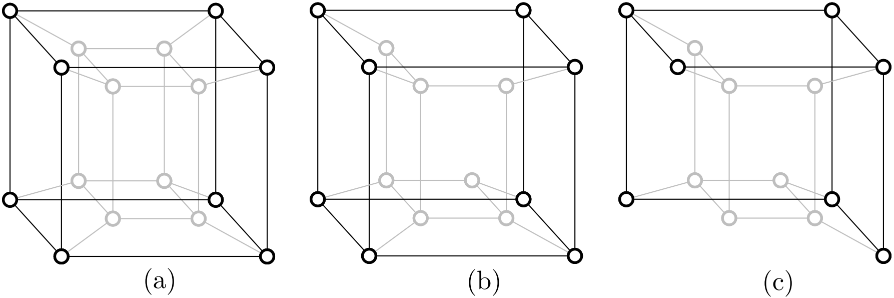

Figure: (a) The 4-cube, a strongly connected 4-complex. (b) A strongly connected 3-complex in the 4-cube. (c) A strongly connected 2-complex in the 4-cube.

- **Ingredient 2:** The Connectivity Theorem holds for cubes.
- **Ingredient 3:** Removing the vertices of any proper face of a cubical d-polytope leaves a “spanning” strongly connected $(d-2)$-complex, and hence a $(d-2)$-connected subgraph.

Ingredient 3 is proved using Ingredient 1.

Connectivity Theorem: Sketch of the proof

Let $0 \leq \alpha \leq d - 3$ and let P be a cubical d-polytope with minimum degree at least $d + \alpha$. Then P is $(d + \alpha)$-connected.

Let X be a minimum separator of the graph $G(P)$ of P, with vertices u and v of P being separated by X.

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- **Claim 1.** If $|X| \leq d + \alpha$ then, for any facet F, the cardinality of $X \cap V(F)$ is at most $d - 1$.

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- **Claim 1.** If $|X| \leq d + \alpha$ then, for any facet F, the cardinality of $X \cap V(F)$ is at most $d - 1$.
- **Claim 2.** If $|X| \leq d + \alpha$ then the set X disconnects at least d facets of P.

Let $0 \leq \alpha \leq d-3$ and let P be a cubical d-polytope with minimum degree at least $d+\alpha$. Then P is $(d+\alpha)$-connected.

Let X be a minimum separator of the graph $G(P)$ of P, with vertices u and v of P being separated by X.

- Suppose $|X| \leq d-1+\alpha$ (Proceeding by contradiction).

Connectivity Theorem: Sketch of the proof (Continued)

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- Suppose $|X| \leq d-1+\alpha$ (Proceeding by contradiction).
- Take a facet F being disconnected by X (it exists by **Claim 2**). Then $|V(F) \cap X| = d-1$ (by **Claim 1**).

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- Removing all the vertices of F from P produces a $(d-2)$-connected subgraph S (by **Ingredient 3**).

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- Removing X doesn't disconnect S (as $|V(S) \cap X| \leq \alpha \leq d - 3$).

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- Removing X doesn't disconnect S (as $|V(S) \cap X| \leq \alpha \leq d - 3$).
- So u can be assumed in F. Every neighbour of u in F is in X (by **Ingredient 1**).
- Since $\deg(u) \geq d + \alpha$, there is a neighbour of u in $V(S) \setminus X$, and u can be linked to v.

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A corollary

Corollary

There are functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ such that, for every d,

1. *the function $f(d)$ gives the maximum number such that every cubical d-polytope with minimum degree $\delta \leq f(d)$ is δ-connected;*
2. *the function $g(d)$ gives the maximum number such that every cubical d-polytope with minimum degree $\delta \leq g(d)$ is δ-connected and whose minimum separator consists of the neighbourhood of some vertex; and*
3. *the functions $f(d)$ and $g(d)$ are bounded from below by $2d - 3$.*

An open problem

An naive exponential bound in d for $f(d)$ is readily available.

Taking the connected sum of two cubical d-polytope P_1 and P_2 with minimum degree δ we can obtain a cubical d-polytope Q with minimum degree δ and a separator of cardinality 2^{d-1}, the number of vertices of the facet along which we glued.

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$$2d-3 \leq g(d) \leq f(d) \leq 2^{d-1}. \tag{1}$$

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$$2d - 3 \leq g(d) \leq f(d) \leq 2^{d-1}. \quad (1)$$

Problem

Provide precise values for the functions f and g or improve the lower and upper bounds in (1).

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Linkedness of cubical polytopes (Hoa & Ugon)

- A graph with at least $2k$ vertices is k-linked if, for every set of $2k$ distinct vertices organised in arbitrary k pairs of vertices, there are k disjoint paths joining the vertices in the pairs.

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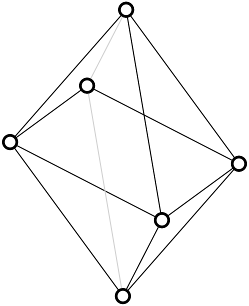
- A k-linked graph is at least $(2k-1)$-connected.

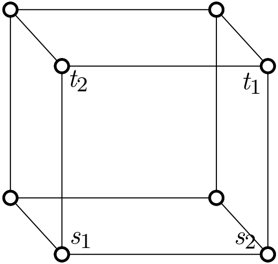
  (If it had a separator X of size $2k-2$, choose k-pairs $(s_1,t_1),\ldots,(s_k,t_k)$ to be linked such that $X := \{s_1,\ldots,s_{k-1},t_1,\ldots,t_{k-1}\}$ and the vertices s_k and t_k are separated by X.)

Classification of 2-linked graphs and 3-polytopes

- (Seymour ’80 and Thomassen ’80) The graph of every simplicial 3-polytopes is 2-linked; that is, every 3-connected planar graph with triangles as faces is 2-linked.
- No other 3-polytope is 2-linked.



(a) 2-linked



(b) not 2-linked

Linkedness of d-polytopes

- (Larman & Mani '70) Every d-polytope is $\lfloor (d+1)/3 \rfloor$-linked.
- (Werner & Wotzlaw '11) Slightly improved to $\lfloor (d+2)/3 \rfloor$.

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- (Thomas & Wollan '05) Every d-polytope with minimum degree at least $5d$ is $\lfloor d/2 \rfloor$-linked.

- (Larman & Mani '70) Graphs of simplicial d-polytopes, polytopes with all its facets being combinatorially equivalent to simplices, are $\lfloor (d+1)/2 \rfloor$-linked.
- This is the maximum possible linkedness given that some of these graphs are d-connected but not $(d+1)$-connected.

- (Wotzlaw '09) In his PhD thesis he asked whether d-cubes were $\lfloor d/2 \rfloor$-linked and whether cubical d-polytopes were $\lfloor d/2 \rfloor$-linked.

Cubical d-polytopes

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- (Meszaros '15) d-cubes are $\lfloor (d+1)/2 \rfloor$-linked for $d \neq 3$. This was as part of a study of linkedness of cartesian products of graphs.

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Theorem (Linkedness Theorem; Hoa, PV & Ugon)

Cubical d-polytopes are $\lfloor (d+1)/2 \rfloor$-linked for every $d \neq 3$.

This is best possible since there are cubical d-polytopes which are d-connected but not $(d+1)$-connected.

An open problem

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Problem (Wotzlaw '09)

Is there some function $h(d)$, such that every d-polytope on at least $h(d)$ vertices is $\lfloor d/2 \rfloor$-linked?

References

- J. Doolittle, E. Nevo, G. Pineda-Villavicencio, J. Ugon and D. Yost, **On the reconstruction of polytopes**, Discrete & Computational Geometry, to appear, arXiv:1702.08739.
- H. T. Bui, G. Pineda-Villavicencio and J. Ugon, **Connectivity of cubical polytopes**, 13 pages, arXiv:1801.06747.
- H. T. Bui, G. Pineda-Villavicencio and J. Ugon, **The linkedness of cubical polytopes**, 29 pages, arXiv:1802.09230.

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